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Inversion for the Radon line transform in higher dimensions

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Contents

Twistor theory is used to develop a new procedure for inverting the Radon line transform in three or more dimensions. First, the inversion problem is reduced to a $\overline{\partial}$-problem on the complement of real projective space in complex projective space. The $\overline{\partial}$-problem in turn is solved by means of an explicit integral formula.

Keywords: Hilbert transform; Liouville's theorem; Radon transform; twistor theory

1. Introduction

The three-dimensional Radon line transform problem (Helgason 1980; Radon 1917) may be formulated as follows. Let F be a given density distribution on Euclidean three space. If λ is a straight line in the space, denote by $\rho_\lambda(F)$ the integral of F along the line λ, using the Euclidean measure along the line. The problem is then: given $\rho_\lambda(F)$, for all lines λ, find the function F. This problem is routine to solve for certain classes of functions F (for example, smooth functions of compact support) using Fourier analysis. Here a new and deeper approach to the problem will be sketched, using complex analysis and twistor theory (Penrose & Rindler 1984, 1986).

The twistor reformulation of the problem was described for the first time by the author in lectures in Oxford in the late 1970s. The reason for the delay in solving it is that the topology and analysis needed are quite subtle and the correct approach only became clear recently. This reformulation will now be presented briefly. For any integer $n \geqslant 2$, consider V, a real vector space of dimension $n+1$, V', the complement of the origin in V and A, a real affine space of dimension n.

Choose an affine embedding $\mu : A \to V'$ of A in V', as a hyperplane in V, not passing through the origin. Denote by $C^\infty(V,-2)$ the space of smooth functions globally defined on the space V' and homogeneous of degree minus two, i.e. if $f \in C^\infty(V,-2)$, then $t^2 f(tv) = f(v)$, for all $v \in V'$ and all non-zero real numbers t.

A real-valued function F defined on the space A will be said to be regular at infinity if and only if $F = f\mu$, for some (necessarily unique) function $f \in C^\infty(V,-2)$. Denote by $G^\infty(A)$ the space of functions F on A that are regular at infinity. The space $G^\infty(A)$ is a subspace of the space of all smooth functions on the space A and is independent of the choice of the embedding μ.

Denote by $T(A) = A \times B$ the tangent bundle of A (where B is a real vector space of dimension n) and by $S(A)$ the complement of the zero section in $T(A)$. For any $x \in A$, $y \in A$ and $v \in B$, we shall write $y - x \in B$ for the position vector of y relative to x and $x + v \in A$ for the point of A whose position vector relative to x is the vector v. The derivative of the map μ is a real linear map $\mu^* : B \to V$, with image the hyperplane through the origin of the space V, parallel to the hyperplane $\mu(A)$.

Asymptotically, we have the limit: $\lim_{t\to\infty} t^2 F(x+tz) = f(\mu^*(z))$, for any $(x,z) \in S(A)$ and for any $F \in G^\infty(A)$, where $F = f\mu$. In particular, for any given $F \in G^\infty(A)$, the following improper integral converges, defining a function $\rho(F)$ on the space $S(A)$:

$$\rho(F)(x,z) \equiv \pi^{-1} \int_{-\infty}^{\infty} F(x+tz)\,\mathrm{d}t.$$

The Radon transform problem for $F \in G^\infty(A)$, is then: given the function $\rho(F)$, retrieve the function F. Note that the quantity $\rho(F)$ is essentially homogeneous of degree minus one in z, in that we have the relation $|k|\rho(F)(x,kz) = \rho(F)(x,z)$, for all $(x,z) \in S(A)$ and for all real $k \neq 0$. Consequently, if the bundle $T(A)$ is provided with an Euclidean structure, knowledge of the function $\rho(F)$ is equivalent to knowledge of its restriction to the subset of $S(A)$ consisting of all $(x,z) \in S(A)$, such that z is a unit vector. In turn the quantity $\rho(F)(x,z)$, for z a unit vector, is a fixed constant multiple of the ordinary line integral of the function F, with respect to Euclidean measure, along the line in the space A through the point x with direction vector z. So knowledge of the function $\rho(F)$, for each $F \in G^\infty(A)$, is equivalent to knowledge of the ordinary Radon transform for $G^\infty(A)$ in the Euclidean case. Note, however, that the quantity $\rho(F)$ does not need an Euclidean structure for its definition.

A prototypical example is the case $F = F_{a,b}$, where $F_{a,b}(x) \equiv [(x-a)\cdot(x-a)+b^2]^{-1}$, for any $x \in A$, any fixed $a \in A$ and any fixed non-zero real number b. Here $u \cdot v$ (for any $u \in B$ and $v \in B$) is an Euclidean inner product for the vector space B.

One finds that $\rho(F_{a,b})(x,z) = [z\cdot z((x-a)\cdot(x-a)+b^2) - (z\cdot(x-a))^2]^{-1/2}$, for any $(x,z) \in S(A)$.

Next denote by $\Delta(V)$ the space of all pairs $(v,w) \in V^2$, with v and w linearly independent. Consider the following integral, for any given $f \in C^\infty(V,-2)$ and for any $(v,w) \in \Delta(V)$:

$$\phi(f)(v,w) \equiv \pi^{-1} \int_{-\pi/2}^{\pi/2} f(v\cos\theta + w\sin\theta)\,\mathrm{d}\theta = (2\pi)^{-1} \int_{-\pi}^{\pi} f(v\cos\theta + w\sin\theta)\,\mathrm{d}\theta.$$

Putting $z \equiv v\cos\theta + w\sin\theta$, we have the following identities, valid for any real θ

and any $(v,w) \in \Delta(V)$:

$$\partial_\theta[-\sin^2\theta f(z)] = v\cdot\partial_w f(z),$$
$$\partial_\theta[\cos^2\theta f(z)] = w\cdot\partial_v f(z),$$
$$\partial_\theta[-\sin\theta\cos\theta f(z)] = v\cdot\partial_v f(z) + f(z),$$
$$\partial_\theta[\sin\theta\cos\theta f(z)] = w\cdot\partial_w f(z) + f(z).$$

Here ∂_θ is the partial derivative with respect to θ and for any vector variables a and b, the operator $a\cdot\partial_b$ is the directional derivative in the direction of a, with respect to the variable b. Integrating these identities immediately gives the relations, valid for all $f \in C^\infty(V,-2)$:

$$v\cdot\partial_w\phi(f) = w\cdot\partial_v\phi(f) = v\cdot\partial_v\phi(f) + \phi(f) = w\cdot\partial_w\phi(f) + \phi(f) = 0.$$

We also have $\phi(f)(-v,w) = \phi(f)(w,v) = \phi(f)(v,w)$, for all $(v,w) \in \Delta(V)$. Combining these relations, we obtain the formula, valid for any real numbers a, b, c and d, such that $ad-bc \neq 0$ and for any $(v,w) \in \Delta(V)$:

$$\phi(f)(av+bw, cv+dw) = |ad-bc|^{-1}\phi(f)(v,w).$$

Next define on the space $\Delta(V)$, a function $\Phi(f)$, taking values in $\Omega^2(V)$, the exterior product of V with itself, by: $\Phi(f)(v,w) \equiv v\wedge w\phi(f)(v,w)$, for any $(v,w) \in \Delta(V)$. Then the function $\Phi(f)$ obeys the following relations, valid for any real numbers a, b, c and d, such that $ad-bc>0$ and for all $(v,w) \in \Delta(V)$:

$$\Phi(f)(v,w) = -\Phi(f)(-v,w) = -\Phi(f)(w,v),$$
$$\Phi(f)(av+bw, cv+dw) = \Phi(f)(v,w).$$

Thus the quantity $\Phi(f)(v,w)$ depends only on the *oriented* two plane in the space V, through the points v and w, with oriented basis (v,w). Denote by $M(V)$ the oriented Grassmanian of oriented two planes in the vector space V. Then the function $\Phi(f)$ may be interpreted as the pullback to the space $\Delta(V)$ of a smooth function, still called $\Phi(f)$, taking values in $\Omega^2(V)$, defined globally on the space $M(V)$. Note that if $x \in M(V)$ and $y \in M(V)$, with $x = y$ as subspaces of V, but with x and y having the opposite orientations, then we have $\Phi(f)(x) = -\Phi(f)(y)$, for all $f \in C^\infty(V,-2)$.

Finally, if $F = f\mu$, for $f \in C^\infty(V,-2)$ and for $F \in G^\infty(A)$, then we have the following relation, valid for all $(x,z) \in S(A)$:

$$\phi(f)(\mu(x), \mu(x+z)) = \rho(F)(x,z).$$

So, knowledge of the function $\phi(f)$ on the space $\Delta(V)$ is equivalent to knowledge of the function $\rho(f)$ on the space $S(A)$, whenever $F = f\mu$. In turn, knowledge of the function $\phi(f)$ on the space $\Delta(V)$ is equivalent to knowledge of the function $\Phi(f)$ on the space $M(V)$. So, solving the Radon transform problem for the function space $G^\infty(A)$ is equivalent to solving the transform problem:

Given the $\Omega^2(V)$-valued function $\Phi(f)$ on the space $M(V)$, for $f \in C^\infty(V,-2)$, reconstruct the function f.

It is in this form that we shall analyse the problem. We shall treat not just the standard three-dimensional Radon transform, for which $n=3$, but the generalization, for which $n \geqslant 3$.

The prototypical example in this language is the case $f = f_g$, where $f_g(v) \equiv [g(v,v)]^{-1}$, for any $v \in V'$. Here the quantity $g(\cdot,\cdot)$, represents a definite symmetric bilinear inner form for the space V. For this example we have, for any $(v,w) \in \Delta(V)$:

$$\phi(f_g)(v,w) = [g(v,v)g(w,w) - g(v,w)^2]^{-1/2}.$$

Pick a unit vector $\alpha \in V$ and an affine embedding μ of A onto the hyperplane with equation $g(\alpha, v) = b \neq 0$ in V. Then we have the relation $f_g\mu = F_{a,b}$, provided $\mu(a) = b\alpha$, where we use the induced metric for the Euclidean structure of the space A. Then the prototypical examples f_g and $F_{a,b}$ are compatible, as is easily seen.

Section 2 below organizes the real and complex geometry underlying the Radon transform. The most important spaces are the spaces V', $M(V)$ and $H(V)$, the space of all complex vectors $a + \mathrm{i}b$, with a and b linearly independent vectors in V. The space $H(V)$ maps to the space $M(V)$, in two ways: one assigns to each vector $a + \mathrm{i}b$ in $H(V)$ the oriented subspace with oriented base (a, b); the other assigns to each vector $a + \mathrm{i}b$ in the space $H(V)$ the oriented subspace with oriented basis (b, a).

Section 3 introduces the differential operator, E, which generates the action of the general linear group of the space V on a variety of associated spaces and discusses applications of Liouville's theorem.

Section 4 gives a precision treatment of the basic Radon transform formula. The transform is described there as a linear operator $\Phi : C^\infty(V, -2) \to Z(M)$, where $Z(M)$ is a space of smooth functions on the space $M(V)$, taking values in $\Omega^2(V)$, subject to a certain system of second-order differential equations. It is shown that these equations, when pulled back to the space $H(V)$, amount to exactly the condition that for each $f \in C^\infty(V, -2)$, a certain $(0, 1)$-form on the space $H(V)$, constructed from the transform $\Phi(f)$, be $\bar{\partial}$-closed. This provides the entry for a complex analytic approach to an otherwise real problem.

Section 5 gives a description of the Hilbert transform, H_X, as a complex structure for the space of smooth functions, homogeneous of degree minus one, defined on the complement of the origin in an oriented two-dimensional real vector space X. Complex analysis enters here, again, since the (complex) eigenvectors of the operator H_X admit holomorphic extensions into certain domains in the complexification of the space X.

Section 6 gives a cohomology vanishing theorem, showing the vanishing of the first cohomology group, of the sheaf $\Theta(-2)$ of germs of sections of the line bundle of Chern class minus two, over the complement of real projective space in complex projective space. This is relevant, since the projective image, $p_{\mathbb{C}}(H)$, of the space $H(V)$ is exactly the complement of the projective space of V inside the complex projective space of the complexification of V.

Section 7 shows how to solve the problem $\bar{\partial}\alpha = \beta$, where $\bar{\partial}$ is the $\bar{\partial}$-operator for complex projective space and β is a $\bar{\partial}$-exact $(0, 1)$ form with coefficients in $\Theta(-2)$, defined on certain domains in projective space. The method used is a global version of the method of Grothendieck. This section gives in particular the solution for the domain $p_{\mathbb{C}}(H)$ in terms of explicit integrals involving the form β.

Section 8 shows how to use the Hilbert transform to get at the Radon transform. In particular, it is shown that the Radon transform Φ is injective.

Section 9 shows how to use the information gathered in §§ 4–8 to solve completely the inversion problem for the Radon transform. Roughly, the procedure followed is to organize the information of the transform $\Phi(f)$, for any $f \in C^\infty(V, -2)$, in a $\bar{\partial}$-closed $(0, 1)$ form, $\beta(f)$, on the space $H(V)$, to prove exactness of the form $\beta(f)$, to solve the equation $\bar{\partial}\alpha(f) = \beta(f)$, with an explicit construction of the (unique) solution $\alpha(f)$ and then to show how to reconstruct the function f from the quantity $\alpha(f)$.

It should be noted that the entire construction is carried out without being able to conclusively identify the range of the transform Φ. It is natural to conjecture,

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however, that the range is the whole space $Z(M)$. Fritz John (1938) wrote a beautiful treatment of the Radon transform for continuous data. Much of his work can be taken over into the present language. It is hoped that by carefully studying his paper it may become possible to settle this conjecture.

This work may be regarded as the third of a series devoted to analysing the relation between twistor theory and soliton theory; the first two works were co-authored with Lionel Mason (Mason & Sparling 1989, 1992). In the language of twistor theory, the present work (specialized to the case that V is four dimensional) deals only with linear spin zero fields obeying the conformally invariant wave equation on the conformally flat manifold $M(V)$, of signature $(2, 2)$. However, the global approach used here is easily seen to be compatible with the standard local twistor generalizations from spin zero to other spins and from linear field theory to (anti)-self-dual connection theory (gauge connections and Cartan conformal connections). This approach then dovetails in nicely with the elegant approach to global soliton theory developed independently by Mason, using non-Hausdorff spaces. Mason's theory was introduced in a seminar by him at the University of Pittsburgh, while the present paper was being written up.

A technical difference is that the author's work considers the smooth theory, whereas Mason's spaces encode most naturally the analytic case. However, it should be remarked that in the nonlinear theory the analysis is far more difficult, so theorems in the present theory have to be replaced (temporarily, one hopes!) by conjectures in the nonlinear case. It is planned to give a fuller discussion of these matters elsewhere.

In the special case that the vector space V is four dimensional, this work gives an alternative to the standard theories of three-dimensional imaging, where the image is to be reconstructed from the observed absorption of linearly propagating probes (sonar, X-rays (Röentgen 1895), neutrino beams, etc.). It is possible that the present approach is superior to the conventional approach. This is currently under investigation.

2. Geometrical organization

Denote by $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{R}^+$, respectively, the real and complex fields and the multiplicative group of positive reals. For any $t \in \mathbb{C}$, $\bar{t}$ will denote its complex conjugate.

For any real vector space V, of finite dimension $d(V) \geqslant 2$, we introduce the following associated entities:

1. $V_{\mathbb{C}}$: the complexification of V. If $v \in V_{\mathbb{C}}$, we write $\bar{v} \in V_{\mathbb{C}}$ for its complex conjugate and $v^+ \in V$ and $v^- \in V$ for its real and imaginary parts, respectively: so $v = v^+ + \mathrm{i}v^-$ and $\bar{v} = v^+ - \mathrm{i}v^-$.
2. V^*, $V^*_{\mathbb{C}}$: the real dual of V and the complex dual of $V_{\mathbb{C}}$.
3. V', $V'_{\mathbb{C}}$: the spaces V and $V_{\mathbb{C}}$ with their origins removed.
4. $T(V)$: the real tensor algebra of V, generated by $\mathbb{R}$, V and V^*.
5. $T^p_q(V)$: the tensor product of p copies of V with q copies of V^*, for p and q non-negative integers.
6. $T^+(V)$, $T^-(V)$: the subalgebras of $T(V)$ generated by $\mathbb{R}$ and V and by $\mathbb{R}$ and V^*, respectively.

7. $T(V_{\mathbb{C}})$: the complex tensor algebra of $V_{\mathbb{C}}$, generated by $\mathbb{C}$, $V_{\mathbb{C}}$ and $V_{\mathbb{C}}^*$.

8. $T_q^p(V_{\mathbb{C}})$: the tensor product of p copies of $V_{\mathbb{C}}$ with q copies of $V_{\mathbb{C}}^*$, for p and q non-negative integers.

9. $T^+(V_{\mathbb{C}})$, $T^-(V_{\mathbb{C}})$: the subalgebras of $T(V_{\mathbb{C}})$ generated by $\mathbb{C}$ and $V_{\mathbb{C}}$ and by $\mathbb{C}$ and $V_{\mathbb{C}}^*$, respectively.

10. $P(V)$, $P_{\mathbb{C}}(V)$: the real projective space of V and the complex projective space of $V_{\mathbb{C}}$.

11. $\Omega(V)$, $\Omega(V_{\mathbb{C}})$: the real exterior algebra of V and the complex exterior algebra of $V_{\mathbb{C}}$, respectively.

12. $\Omega^p(V)$, $\Omega^p(V_{\mathbb{C}})$: the p-fold exterior powers of V and $V_{\mathbb{C}}$, respectively, for any non-negative integer p.

We use $\wedge$ to denote the exterior products of the algebras $\Omega(V)$ and $\Omega(V_{\mathbb{C}})$, not for the exterior product for differential forms; instead this latter product is written as juxtaposition of the terms in the product. For any given $a \in V^*$, $\alpha \in \Omega(V)$, $b \in V_{\mathbb{C}}^*$, $\beta \in \Omega(V_{\mathbb{C}})$, denote by $\iota(a)\alpha \in \Omega(V)$ and $\iota(b)\beta \in \Omega(V_{\mathbb{C}})$, respectively, the images of α and β under the natural actions of a and b as derivations of $\Omega(V)$ and $\Omega(V_{\mathbb{C}})$ of degree minus one. We occasionally identify $\Omega^1(V)$ and V, without comment.

1. $M(V)$: the space of *oriented* two-dimensional real subspaces of V.

2. $N(V)$: the space of all $X \in \Omega^2(V)$, such that $0 \neq X$ and $X \wedge X = 0$.

3. $\Delta(V)$: the space of all $(v, w) \in V^2$, such that $v \wedge w \neq 0$.

4. $H(V)$: the space of all $v \in V_{\mathbb{C}}$, such that $v \wedge \overline{v} \neq 0$.

5. $S(V)$: the space of all pairs (x, v), with $x \in M(V)$ and $v \in x'$.

6. $S_{\mathbb{C}}(V)$: the space of all pairs (x, v), with $x \in M(V)$, $v \in V_{\mathbb{C}}'$, $v^+ \in x$ and $v^- \in x$.

7. $S_+(V)$: the space of all $(x, v) \in S_{\mathbb{C}}(V)$, such that (v^+, v^-) is an oriented basis of x.

8. $S_-(V)$: the space of all $(x, v) \in S_{\mathbb{C}}(V)$, such that $(v^+, -v^-)$ is an oriented basis of x.

9. $S_0(V)$: the space of all $(x, v) \in S_{\mathbb{C}}(V)$, such that $v \wedge \overline{v} = 0$.

Note that the space $S_{\mathbb{C}}(V)$ is the disjoint union of the spaces $S_+(V)$, $S_-(V)$ and $S_0(V)$ and the space $S_0(V)$ is the boundary in $S_{\mathbb{C}}(V)$ of each of the spaces $S_+(V)$ and $S_-(V)$. The space $S(V)$ is a subspace of the space $S_0(V)$ and if (x, v) is a given point of $S_0(V)$, then there exists a point (x, w) of $S(V)$, such that $v = tw$, for some $t \in \mathbb{C}$ with $t\overline{t} = 1$ and then the vector w is unique up to sign.

We also introduce the following maps.

1. The natural skew symmetrization maps:

$$\begin{aligned}\wedge^+(V) : T(V) &\to T^-(V) \otimes \Omega(V),\\ \wedge^+(V_{\mathbb{C}}) : T(V_{\mathbb{C}}) &\to T^-(V_{\mathbb{C}}) \otimes \Omega(V_{\mathbb{C}}),\\ \wedge^-(V) : T(V) &\to T^+(V) \otimes \Omega(V^*),\\ \wedge^-(V_{\mathbb{C}}) : T(V_{\mathbb{C}}) &\to T^+(V_{\mathbb{C}}) \otimes \Omega(V_{\mathbb{C}}^*),\\ \wedge(V) : T(V) &\to \Omega(V) \otimes \Omega(V^*),\\ \wedge(V_{\mathbb{C}}) : T(V_{\mathbb{C}}) &\to \Omega(V_{\mathbb{C}}) \otimes \Omega(V_{\mathbb{C}}^*).\end{aligned}$$

2. The canonical projections:

$$\begin{aligned}p(V) : V' &\to P(V),\\ p_{\mathbb{C}}(V) : V'_{\mathbb{C}} &\to P_{\mathbb{C}}(V).\end{aligned}$$

3. $m(V) : N(V) \to M(V)$, the canonical surjection, which assigns to any given $X \in N(V)$ the oriented subspace of V, $m(V)(X) \in M(V)$, with oriented basis the ordered pair of vectors (v, w) of V, whenever we have $v \wedge w = X$.
4. $n(V) : \varDelta(V) \to N(V)$, $(v, w) \in \varDelta(V) \to n(V)(v, w) \equiv v \wedge w \in N(V)$.
5. $X_+(V) : H(V) \to N(V)$, $v \in H(V) \to X_+(V)(v) \equiv 2v^+ \wedge v^- \in N(V)$.
6. $X_-(V) : H(V) \to N(V)$, $v \in H(V) \to X_-(V)(v) \equiv -2v^+ \wedge v^- \in N(V)$.
7. $x_+(V) : H(V) \to M(V)$, the composition $m(V)X_+(V)$.
8. $x_-(V) : H(V) \to M(V)$, the composition $m(V)X_-(V)$.
9. $s_+(V) : H(V) \to S_+(V)$, $v \in H(V) \to s_+(V)(v) \equiv (x_+(V)(v), v) \in S_+(V)$.
10. $s_-(V) : H(V) \to S_-(V)$, $v \in H(V) \to s_-(V)(v) \equiv (x_-(V)(v), v) \in S_-(V)$.

As a subspace of V, we have $m(V)(X) = \{v \in V; v \wedge X = 0\}$, for any $X \in N(V)$. Then we have $m(V)(X) = m(V)(Y)$, for $X \in N(V)$ and $Y \in N(V)$, if and only if there exists a unique $r \in \mathbb{R}^+$, with $X = rY$. The map $m(V)$ makes $N(V)$ into a principal fibre bundle over the space $M(V)$ with fibre $\mathbb{R}^+$.

If the underlying vector space V is understood, we shall abbreviate, when it is convenient to do so, by omitting (V) from the names of the various spaces and maps introduced above and by replacing $(V_{\mathbb{C}})$ and $(V_{\mathbb{C}}^*)$ by just $_{\mathbb{C}}$ and $^*_{\mathbb{C}}$, respectively.

Lemma 2.1. *The maps s_+ and s_- are diffeomorphisms, so provide the spaces S_+ and S_-, respectively, with complex structures.*

Proof. The maps s_+ and s_- are clearly smooth and are easily seen to be injective and surjective. ■

If W is a subspace of V, denote by $\mathrm{ann}(W)$ the subspace of V^* consisting of all linear forms on V that annihilate the subspace W. For any W, a subspace of V, denote by $\sigma(W)$ the subalgebra of the algebra T generated by the spaces $\mathbb{R}$, W and $\mathrm{ann}(W)$ and by $\sigma^p_q(W)$ the intersection of the spaces $\sigma(W)$ and T^p_q, for any p and q non-negative integers.

By definition, the twistor algebra over the space M, $\tau(M)$ is the trivial tensor algebra bundle $M \times T$.

By definition, the spinor algebra over the space M, $\sigma(M)$, is the algebra subbundle of the twistor algebra over M, whose fibre at any point $x \in M$ is the algebra $\sigma(x)$. A section of the twistor algebra that takes values in $\sigma(M)$ will be called local, or a spinor field. For any p and q non-negative integers, denote by σ^p_q the bundle over M, whose fibre at any $x \in M$ is the space $\sigma^p_q(x)$. For any p and q non-negative integers, a spinor field which is a section of σ^p_q will be said to be of type (p, q) or to have p primed indices and q unprimed indices.

Note that the space $S(V)$ introduced above is the complement of the zero section of the primed spin bundle σ^1_0.

Some quantities are most easily expressed using indices. We shall use Greek lower-case indices for the tensor algebras of V and $V_{\mathbb{C}}$, generally conforming to the abstract index conventions of Penrose & Rindler (1984).

3. Canonical differential operators; derivations; homogeneous functions; fibre holomorphic functions; Liouville's theorem

We work in the smooth category throughout. Denote by d the exterior derivative operator, for any space. Denote by $\overline{\partial}$ the $\overline{\partial}$-bar operator for any complex manifold. One has $\mathrm{d} = \partial + \overline{\partial}$, where ∂ is the complex conjugate of $\overline{\partial}$.

For any vector field ρ, denote by $\iota(\rho)$ the corresponding derivation of forms of degree minus one and by $L(\rho)$ the Lie derivative derivation of forms, in the direction of ρ. One has the relation $L(\rho) = \mathrm{d}\iota(\rho) + \iota(\rho)\,\mathrm{d}$.

Denote by z the tautological V-valued function on V, whose value at any $v \in V$ is just v. The function z may be used as a (vector) coordinate function for V. Denote by ∂_z the V^* valued vector field on V, such that $(\partial_z)(z) = \delta$, where δ is the Kronecker delta tensor.

Denote by ζ (with complex conjugate $\overline{\zeta}$) the tautological $V_{\mathbb{C}}$-valued function on $V_{\mathbb{C}}$, whose value at any $v \in V_{\mathbb{C}}$ is v. The function ζ may be used as a (vector) holomorphic coordinate function for $V_{\mathbb{C}}$. Denote by ∂_ζ (with complex conjugate $\partial_{\overline{\zeta}}$) the $V^*_{\mathbb{C}}$ valued vector field on $V_{\mathbb{C}}$, such that $(\partial_\zeta)(\zeta) = \delta$ and $(\partial_\zeta)(\overline{\zeta}) = 0$.

If $t \in \mathbb{R}$, denote by $\varGamma(t)$ the diffeomorphism $\varGamma(t) : V' \to V'$, $v \to e^t v$, for all $v \in V'$. The set $\varGamma \equiv \{\varGamma(t); t \in \mathbb{R}\}$ forms a one-parameter group of diffeomorphisms of V'. Denote by η the vector field generating $\varGamma$. Explicitly, we have the relation $\eta(f) = \iota(z)(\partial_z(f))$, for any smooth function f on V' (not necessarily globally defined). A quantity defined on V' will be said to be homogeneous of (integral) degree k, if and only if, for each real t, it scales by a factor of e^{tk}, under the action of the diffeomorphism $\varGamma(t)$. Similar definitions apply to functions defined on open subsets of V' that are invariant under the action of the group $\varGamma$. For any integer k, denote by $C^\infty(V, k)$ the space of all smooth functions f, globally defined on V' and homogeneous of degree k, such that $f(-v) = (-1)^k f(v)$, for all $v \in V'$.

For each complex t, denote by $\varGamma_{\mathbb{C}}(t)$ the diffeomorphism $\varGamma_{\mathbb{C}}(t) : V'_{\mathbb{C}} \to V'_{\mathbb{C}}$, $v \to e^t v$, for all $v \in V'_{\mathbb{C}}$. Then the set $\varGamma_{\mathbb{C}} \equiv \{\varGamma_{\mathbb{C}}(t); t \in \mathbb{C}\}$ forms a one-parameter group of holomorphic diffeomorphisms of $V'_{\mathbb{C}}$. A quantity defined on the space V' will be said to be homogeneous of (integral) degrees (p, q), if and only if, for each complex t, it scales by a factor of $\exp(tp + \overline{t}q)$, under the action of $\varGamma_{\mathbb{C}}(t)$. If the quantity is holomorphic and homogeneous of degree (p, q), then necessarily $q = 0$. In that case we shall say that the quantity is homogeneous of degree p. Similar definitions apply to functions defined on open subsets of $V'_{\mathbb{C}}$ that are invariant under the action of $\Gamma_{\mathbb{C}}$.

Denote by G the Lie group of isomorphisms of V and by g the Lie algebra of G. Then the group G acts naturally as a group of diffeomorphisms on each of the spaces V, V^*, Ω^2, $\Omega^2 \times V_{\mathbb{C}}$, $V_{\mathbb{C}}$, P, $P_{\mathbb{C}}$, M, N, H, S, $S_{\mathbb{C}}$, $S_{\pm}$ and S_0. Correspondingly the Lie algebra g is naturally represented on each of these spaces by a global g^*-valued vector field. Denote this vector field, for any of these spaces by E. Note that both g and g^* may be canonically identified with $V \otimes V^*$, so that E may be regarded as taking values in $V \otimes V^*$. Explicit formulas for E follow.

1. On the space V, we have $E = z \otimes \partial_z$.
2. On the space $V_{\mathbb{C}}$, we have $E = \zeta \otimes \partial_\zeta + \overline{\zeta} \otimes \partial_{\overline{\zeta}}$.
3. A point of the space Ω^2 is represented canonically by a skew coordinate vector $X^{\alpha\beta}$. Then the equations of the subspace N are: $X^{\alpha\beta} \neq 0$ and $X^{[\alpha\beta}X^{\gamma\delta]} = 0$. Denote by $\partial_{\alpha\beta}$ the indexed vector field on the space Ω^2, which satisfies the equations $\partial_{\alpha\beta}X^{\gamma\delta} = \delta^{\gamma}_{[\alpha}\delta^{\delta}_{\beta]}$. Then on the space Ω^2, we have the formula $E^{\alpha}_{\beta} = 2X^{\alpha\gamma}\partial_{\beta\gamma}$. The vector field E of the space Ω^2 is tangent to the submanifold N, inducing the vector field E of the space N. The vector field E is invariant under the action of the group $\mathbb{R}^+$ on the space N, so passes down to the space M, giving the vector field E of that space.
4. On the space $\Omega^2 \times V_{\mathbb{C}}$, the vector field E is given by the formula $E^{\beta}_{\alpha} = 2X^{\beta\gamma}\partial_{\alpha\gamma} + \zeta^{\beta}\partial_\alpha + \overline{\zeta}^{\beta}\overline{\partial}_\alpha$, where $\partial_\alpha \equiv (\partial_\zeta)_\alpha$ and $\overline{\partial}_\alpha \equiv (\partial_{\overline{\zeta}})_\alpha$. This vector field restricts naturally to the space $N \times V_{\mathbb{C}}$, passes down to the quotient $M \times V_{\mathbb{C}}$ and is tangent to the submanifold $S_{\mathbb{C}}$ of the space $M \times V_{\mathbb{C}}$.

The fibre of the space $S_{\mathbb{C}}$ at any point $x \in M$ is by definition the set of all points $(x, v) \in S_{\mathbb{C}}$, with $v \in V'_{\mathbb{C}}$ and $v^{\pm} \in x$. If f is a function defined on an open subset of the space $S_{\mathbb{C}}$, then f will be said to be fibre holomorphic if and only if the restriction of f to any fibre is holomorphic with respect to the complex structure on the fibre induced from the complex structure of $V'_{\mathbb{C}}$. Expressed in terms of differential equations on the space $N \times V_{\mathbb{C}}$, the function f is fibre holomorphic if and only if it obeys the relations: $\zeta^\alpha\overline{\partial}_\alpha f = \overline{\zeta}^\alpha\overline{\partial}_\alpha f = X^{\alpha\beta}\partial_{\alpha\beta} f = 0$.

The vector field E on the space M is local, taking values in the bundle σ^1_1. Then the map $\alpha \to \sigma(\alpha) \equiv \iota(E)(\alpha)$, for any α a section of the cotangent bundle of M, gives a natural isomorphism of the cotangent bundle of M, which is a $2(d(V) - 2)$-dimensional bundle over M, with the bundle σ^1_1, which has the same dimension. A metric or conformal structure γ on the space M, will be said to be of spinor type, if and only if the inner products $\gamma(E^{\rho}_{\alpha}, E^{\sigma}_{\beta})$ are totally skew. In the case that the vector space V is four dimensional, it is well known that there is a unique conformal structure γ, of spinor type, for the space M. Distinct points x and y of the space M are connected by a unique null geodesic of that conformal structure, if and only if the intersection of the subspaces x and y of V is exactly one dimensional.

If ω is a differential k-form on M, for k a non-negative integer, then ω may be represented as a spinor by the quantity $\sigma(\omega) \equiv (k!)^{-1}((\iota(E)\otimes)^k(\omega)$. The quantity $\sigma(\omega)$ is a (k, k) spinor field, such that it changes sign whenever pairs of corresponding contravariant and covariant indices are simultaneously interchanged. Conversely, any such (k, k) spinor field represents a unique k-form on M. In particular, if ω is a two form on M, it is represented by a $(2, 2)$ spinor. There is then a canonical two form ω^*, called the dual of ω, which is represented by the spinor $\sigma(\omega)$ with its contravariant indices interchanged. One has $(\omega^*)^* = w$, for any two form w. The two form ω is said to be self-dual if and only if $\omega = \omega^*$ and anti-self-dual if and only if $\omega = -\omega^*$. The bundle of two forms of the space M, which has dimension $(d(V) - 2)(2d(V) - 5)$, is the direct sum of the bundles of self-dual two forms, of dimension $3(d(V)-2)(d(V)-3)/2$ and the bundle of anti-self-dual two forms, which has dimension $(d(V) - 2)(d(V) - 1)/2$.

Next denote by δ the degree zero derivation of the algebra W, taking values in $V \otimes V^*$, representing the natural action of the Lie algebra g on the algebra Ω. We have the explicit formula: $[(\iota(a)\delta)(b)](\omega) = b \wedge (\iota(a)(\omega))$, for any $a \in V^*$, $b \in V$ and $\omega \in \Omega$. Also denote by $\delta_{\mathbb{C}}$ the degree zero derivation of $\Omega_{\mathbb{C}}$, taking values in $V_{\mathbb{C}} \otimes V_{\mathbb{C}}^*$, given by the analogous formula: $[(\iota(e)\delta_{\mathbb{C}})(f)](\omega) = f \wedge (\iota(e)(\omega))$, for any $e \in V_{\mathbb{C}}^*$, $f \in V_{\mathbb{C}}$ and $\omega \in \Omega_{\mathbb{C}}$. Note that acting on the spaces Ω^1 and $\Omega^1_{\mathbb{C}}$, respectively, the derivations δ and $\delta_{\mathbb{C}}$ agree with the action of the real and complex Kronecker delta tensors, respectively.

We shall frequently use variants of Liouville's theorem. The simplest version needed here is as follows. Let D_0 denote a circle on the Riemann sphere, S, and let $D_\pm$ denote open discs, each with boundary D_0, such that the union of $D_\pm$ and D_0 is the whole sphere. Let $f_\pm$ be holomorphic functions on $D_\pm$ that each possess smooth extensions to the closure of their respective domains. Suppose that the restrictions to D_0 of the extensions of $f_\pm$ are equal. Then the functions $f_\pm$ are each everywhere constant, with the same constant value. This result follows because the conditions on the functions $f_\pm$ guarantee (via the Cauchy–Riemann equations) the existence of a global holomorphic function f on S, whose restrictions to the discs $D_\pm$ are the given functions $f_\pm$. But by the standard Liouville theorem, such a function f must be constant, as required. We also need a 'twisted' version of this result. For any integer k, let $\Theta_S(k)$ denote the sheaf of germs of holomorphic sections of the holomorphic line bundle over S of Chern class k (so the Hopf bundle corresponds the case $k = -1$). Let $s_\pm$ be holomorphic sections of $\Theta_S(k)$ on $D_\pm$, that each possess smooth extensions to the closure of their respective domains. Suppose that the restrictions to D_0 of the extensions of the sections $s_\pm$ are equal. Then if $k < 0$, the sections are everywhere zero. This result follows from the untwisted case. First, the conditions on the sections $s_\pm$ guarantee the existence of a global holomorphic section s of $\Theta_S(k)$, whose restrictions to the subsets $D_\pm$ are the given sections $s_\pm$. Second, if $k < 0$, for any global section t of $\Theta_S(1)$, the product $t^{-k}s$ represents a global holomorphic function on S, so is a constant, c, say, by the standard Liouville's theorem. But it is well known that any global section of $\Theta_S(1)$ vanishes at exactly one point of S, so t vanishes at a unique point $p(t)$ of S. Evaluating the product $t^{-k}s$ at $p(t)$ shows that the constant c vanishes. Then evaluating the product at any $p \in S$, with $p \neq p(t)$ gives $s(p) = 0$. So the section s vanishes everywhere on the space S, except possibly at the point $p(t)$. Then by continuity, the section s vanishes everywhere on the space S, as required.

One may cast these facts in the concrete language of homogeneous functions as follows. Let x be an oriented two-dimensional real vector space. Denote by $D_\pm(x)$ the set of all $v \in x'_{\mathbb{C}}$, such that $v^+, \pm v^-$ is an oriented basis of x. Denote by $D_0(x)$ the common boundary of the spaces $D_\pm(x)$ in the space $x'_{\mathbb{C}}$. So $D_0(x) = \{v \in x'_{\mathbb{C}}; v \wedge \overline{v} = 0\}$.

Let $f_\pm$ be smooth functions defined on the closures of $D_\pm(x)$ in $x'_{\mathbb{C}}$ and holomorphic on $D_\pm(x)$. Suppose that the restrictions of $f_\pm$ agree on the boundary $D_0(x)$. Suppose also that the functions $f_\pm$ are everywhere homogeneous of degree k, for some integer k. Then if $k < 0$, the functions $f_\pm$ vanish identically and if $k = 0$, the functions $f_\pm$ are everywhere the same constant.

4. The Radon transform formula

Denote by θ the Ω^1-valued one form on V, such that for any fixed $a \in V^*$, we have: $\iota(a)(\theta) = \mathrm{d}(\iota(a)(z))$. We may trivially extend the exterior derivative of V to act on forms on V, with values in the algebras T or Ω. Then we have $\theta = \mathrm{d}z$. Now the two form θ^2 takes values in Ω^2. Denote by ω the one form $\frac{1}{2}\iota(\eta)(\theta^2)$. Then ω is a globally defined one form, homogeneous of degree two on the space V and taking values in the space Ω^2. Explicitly we have the relation $\omega = z \wedge \theta$. Note that the form ω has exterior derivative $\mathrm{d}\omega = \theta^2$.

For any $x \in M$ denote by x_* the pullback map, restricting forms on V (not necessarily globally defined) to the subspace x. If $v \in x \in M$, then $v \wedge x_*(w) = 0$ and $v \wedge x_*(\theta^2) = 0$. In particular, we have: $x_*(z \wedge \theta^2) = 0$, for all $x \in M$. Let a function $f \in C^\infty(V, -2)$ be given. Then the product fw is a one form, globally defined on V' and homogeneous of degree zero, taking values in the space Ω^2.

Lemma 4.1. *The exterior derivative of the form $f\omega$ is given by the following formula:*

$$\mathrm{d}(fw) = -\tfrac{1}{2}(\iota(\partial_z f)(z \wedge \theta^2)).$$

Proof. We have, using the relations $L(\eta)\theta = \theta$, $L(\eta)f = -2f$ and $\mathrm{d}\theta = 0$:

$$\begin{aligned} 2\mathrm{d}(f\omega) &= \mathrm{d}(f\iota(\eta)(\theta^2)) = \mathrm{d}\iota(\eta)(f\theta^2) = -\iota(\eta)\mathrm{d}(f\theta^2) + L(\eta)(f\theta^2) \\ &= -\iota(\eta)[(\mathrm{d}f)(\theta^2)] = -\iota(\eta)[(\iota(\partial_z f)(\theta))(\theta^2)] \\ &= -\tfrac{1}{3}\iota(\eta)[\iota(\partial_z f)(\theta^3)] = -\iota(\partial_z f)(z \wedge \theta^2), \end{aligned}$$

as required. ∎

Corollary 4.2. *For all $x \in M$, the form $x_*(f\omega)$ is closed.*

Proof. We have

$$2\mathrm{d}[x_*(f\omega)] = 2x_*(\mathrm{d}(f\omega)) = -x_*(\iota(\partial_z f)(z \wedge \theta^2)) = 0,$$

since $x_*(z \wedge \theta^2) = 0$. ∎

By corollary 4.2, the one form $x_*(f\omega)$ defines an element of the cohomology class $H^1(x', \Omega^2)$. This element will be denoted by $[x_*(f\omega)]$. Note that for any $v \in x \in M$, we have $v \wedge [x_*(f\omega)] = 0$, since $v \wedge x_*(w) = 0$.

Now if X is any *oriented* two-dimensional vector space, since X' is homotopic to an oriented circle, the cohomology group $H^1(X', A)$ is canonically isomorphic to A, for any abelian group of (constant) coefficients A. Denote by $\int_X$ the canonical isomorphism $\int_X : H^1(X', A) \to A$, for any A.

Definition 4.3. The Radon transform of any function $f \in C^\infty(V,-2)$ is the function $\Phi(f) \in C^\infty(M,\Omega^2)$, defined by the following formula, for any $x \in M$:

$$\Phi(f)(x) \equiv (2\pi)^{-1}\int_x [x_*(f\omega)]. \tag{4.1}$$

The Radon transform is the linear operator $\Phi : C^\infty(V,-2) \to C^\infty(M,\Omega^2)$, taking each $f \in C^\infty(V,-2)$ to its Radon transform $\Phi(f)$.

The standard twistor Radon problem may now be stated succinctly as follows: if the space V has dimension four:

1. determine the range of the operator Φ,
2. show that the operator Φ is an isomorphism onto its range, and
3. find an explicit inverse for Φ.

In this work parts 2 and 3 of this problem are solved, but not part 1. We allow the space V to have any finite dimension greater than three.

We may write out equation (4.1) more explicitly, as follows: if we pull back the function $\Phi(f)$ along the map mn, to the space Δ, then the function $\Phi(f)$ becomes a function of a pair of vector variables (v,w), with $v \in V$, $w \in V$, $v \wedge w \neq 0$ and $x = mn(v,w)$. Then the circle of points of x', given by $v\cos\theta + w\sin\theta$, with $\theta \in [-\pi,\pi]$, with its natural orientation, represents the positive generator of the first integral homology group of the space x'. The pullback of the form $x_*(w)$ to this circle is the form $v \wedge w\,\mathrm{d}\theta$. Then equation (4.1) gives the following formula:

$$\Phi(f)(v,w) = (2\pi)^{-1} v \wedge w \int_{-\pi}^{\pi} f(v\cos\theta + w\sin\theta)\,\mathrm{d}\theta. \tag{4.2}$$

So the function $\Phi(f)$ defined by formula (4.1) exactly agrees with the function $\Phi(f)$ as discussed in the introduction.

Now the function $\Phi(f)$ obeys a second-order differential equation. To see this, first extend the vector field E to act on $C^\infty(M,\Omega^2)$, by requiring that E act trivially on Ω^2. Direct calculation gives $E(\Phi(f)) = \Phi(E(f)) + \delta(\Phi(f))$. Note that the operator E preserves homogeneity, so maps the space $C^\infty(V,-2)$ to the space $V \otimes V^* \otimes C^\infty(V,-2)$. Extend the derivation d to act on the space $\Omega \otimes T$, by giving it the trivial action on the algebra T. Also put $F \equiv E - \delta$, so that we now have the equation $F(\Phi(f)) = \Phi(E(f))$, for any $f \in C^\infty(V,-2)$.

Now the quantity $F(\Phi(f))$ is a smooth function on M taking values in the space $T_1^1 \otimes \Omega^2$, so also may be regarded as a function taking values in the space T_1^3, skew in its last two indices. Denote by $\wedge F(\Phi(f)) \in C^\infty(M, \Omega^1(V^*) \otimes \Omega^3)$ the composition of the function $F(\Phi(f)) \in C^\infty(M,T)$ with the skew symmetrization map $\wedge$.

Lemma 4.4. *For any $f \in C^\infty(V,-2)$, we have the following identity:*

$$\wedge F(\Phi(f)) = 0. \tag{4.3}$$

Proof. At any $x \in M$, we have

$$\wedge F(4\pi\Phi(f))(x) = \wedge(4\pi\Phi(E(f))(x) = \wedge\left(\int_x [2x_*(E(f)\omega)]\right)$$
$$= \wedge\left(\int_x [x_*(\partial_z f) \otimes z \otimes (z \otimes \theta - \theta \otimes z)]\right) = 0.$$

Here we have used the relation $\wedge(z \otimes (z \otimes \theta - \theta \otimes z)) = 2z \wedge z \wedge \theta = 0$. ■

Equation (4.3) amounts to a kinematic equation (see the remarks after equation (4.9) below). We need to work harder to get the required field equation.

Denote the operation of interchanging the pth and qth indices of a tensor by S^{pq}, for contravariant indices and by S_{pq} for covariant indices. Now, on quantities defined on the space V, the second-order differential operator $E\otimes E$ acts as the operator $E\otimes E = z\otimes\partial_z\otimes z\otimes\partial_z = z\otimes\delta\otimes\partial_z + z\otimes z\otimes\partial_z\otimes\partial_z$. Then, applying S^{12} to both sides of this formula, we get: $S^{12}(E\otimes E) = \delta\otimes z\otimes\partial_z + z\otimes z\otimes\partial_z\otimes\partial_z = \delta\otimes E + z\otimes z\otimes\partial_z\otimes\partial_z$. In particular, we see from this relation that the operator $S^{12}(E\otimes E) - \delta\otimes E$ is totally symmetric. Thus the field $\Phi(f)$ obeys the field equation that the quantity $[S^{12}(F\otimes F) - \delta\otimes F]\Phi(f)$, regarded as taking values in the space $T_2^2\otimes\Omega^2$ is totally symmetric in its tensor arguments.

Denote by $U(\Phi)$ the quantity $[S^{12}(F\otimes F) - \delta\otimes F]\Phi(f)$, regarded as a function on M, taking values in the space $T_2^2\otimes\Omega^2$. Then we have the decomposition $U(\Phi) = U_1^1(\Phi) + U_{-1}^1(\Phi) + U_1^{-1}(\Phi) + U_{-1}^{-1}(\Phi)$, where the quantities $U_q^p(\Phi)$, for $p^2 = q^2 = 1$ obey the relations $S^{12}(U_q^p(\Phi)) = pU_q^p(\Phi)$ and $S^{12}(U_q^p(\Phi)) = qU_q^p(\Phi)$. However, the quantities $U_{-1}^1(\Phi)$ and $U_1^{-1}(\Phi)$ vanish identically because the operators E represent the action of the Lie algebra g. Thus we have just $U(\Phi) = U_1^1(\Phi) + U_{-1}^{-1}(\Phi)$.

So the condition that $U(\Phi)$ be totally symmetric, i.e. that $U(\Phi) = U_1^1(\Phi)$, is equivalent to the condition that the quantity $U_{-1}^{-1}(\Phi)$ vanishes. Therefore, the field Φ obeys the field equation that the completely skew part of the quantity $[S^{12}(F\otimes F) - \delta\otimes F]\Phi(f)$ should vanish, or equivalently that the totally skew part of $(F\otimes F + \delta\otimes F)\Phi(f)$ must vanish, so that $\wedge(F\otimes F+\delta\otimes F)\Phi(f) = 0$. Note that $\wedge(F\otimes F+\delta\otimes F)\Phi(f)$ is an element of $\Omega\otimes\Omega(V^*)\otimes C^\infty(M,\Omega^2)$.

We may summarize the above discussion with definition 4.5 and lemma 4.6 as follows.

Definition 4.5. The linear space $Z(M)$ is the space of all smooth Ω^2-valued functions Ψ, defined globally on the space M, and obeying the following relations, valid for any $v \in x \in M$:

$$v\wedge\Psi(x) = 0; \tag{4.4}$$

$$\wedge F(\Psi)(x) = 0; \tag{4.5}$$

$$\wedge(F\otimes F + \delta\otimes F)(\Psi)(x) = 0. \tag{4.6}$$

Lemma 4.6. *The map F has range in the space $Z(M)$.*

A separate proof of lemma 4.6 is given below (lemma 8.4). Because of lemma 4.6, we henceforth regard the linear operator F as a linear map, $F: C^\infty(V,-2)\to Z(M)$.

We next state the main conjecture.

Conjecture 4.7. *The map $F: C^\infty(V,-2)\to Z(M)$ is an isomorphism and its inverse may be explicitly given.*

The purpose of the present work is to prove, using complex analysis, the following result.

Theorem 4.8. *The map $F: C^\infty(V,-2)\to Z(M)$ is an isomorphism on to its range and its inverse may be explicitly given.*

A precise statement and proof of theorem 4.8 is provided in theorem 9.2 below.

To proceed further we need to simplify the field equations. Using indices the desired equations, given in definition 4.5, for a field Ψ, defined globally on M and taking values in Ω^2 are the following three equations, valid for all $v \in x \in M$:

$$v^{[\alpha}\Psi^{\beta\gamma]}(x) = 0; \tag{4.7}$$

$$E^{[\alpha}_{\delta}\Psi^{\beta\gamma]}(x) + 2\delta^{[\alpha}_{\delta}\Psi^{\beta\gamma]}(x) = 0; \tag{4.8}$$

$$\begin{aligned} E^{[\alpha}_{[\gamma}E^{\beta]}_{\delta]}\Psi^{\epsilon\zeta}(x) + 4E^{[\alpha}_{[\gamma}\Psi^{\beta][\epsilon}(x)\delta^{\zeta]}_{\delta]} + \delta^{[\alpha}_{[\gamma}E^{\beta]}_{\delta]}\Psi^{\epsilon\zeta}(x) \\ + 4\delta^{[\alpha}_{[\gamma}\Psi^{\beta][\epsilon}(x)\delta^{\zeta]}_{\delta]} + 2\delta^{[\epsilon}_{[\gamma}\delta^{\zeta]}_{\delta]}\Psi^{\alpha\beta}(x) = 0. \end{aligned} \tag{4.9}$$

Here the field $\Psi^{\alpha\beta}$ is skew in the indices α and β. First, pull back the field Ψ to the space N, along the map m. Then equation (4.7) holds if and only if we have the decomposition $\Psi = X\psi$, where X is the Ω^2-valued coordinate function of the space Ω^2 restricted to the space N and ψ is a scalar field, globally defined on N, homogeneous of degree minus one in the variable X. Then calculating the derivative $E(\Psi)$, we get $E(\Psi) = \delta(X)\psi + XE(\psi)$, or $F(\Psi) = XE(\psi)$. Then we have

$$(\wedge F(\Psi))^{\alpha\beta\gamma}_{\delta} = 2X^{[\alpha\beta}X^{\gamma]\epsilon}\partial_{\delta\epsilon}\psi = 0.$$

Therefore, given equation (4.7), equation (4.8) follows automatically, confirming the statement above that equation (4.3) is only a kinematic equation. Equation (4.9) now reduces to the simpler equation $\wedge(E\otimes E + \delta\otimes E)\psi = 0$. Using indices this field equation is the following:

$$E^{[\alpha}_{[\gamma}E^{\beta]}_{\delta]}\psi + \delta^{[\alpha}_{[\gamma}E^{\beta]}_{\delta]}\psi = 0. \tag{4.10}$$

Next we pull the field ψ back to the space Δ along the map n, via the relation $X = v\wedge w$, with $(v, w) \in \Delta$. Then the operator E decomposes as $E = v\otimes\partial_v + w\otimes\partial_w$, where (v, w) are the V-valued coordinate functions for $V\times V$ and ∂_v and ∂_w are the corresponding V^*-valued coordinate vector fields. This entails the following relation:

$$\begin{aligned} \wedge(E\otimes E) &= -\wedge[S^{12}(v\otimes\partial_v\otimes v\otimes\partial_v + v\otimes\partial_v\otimes w\otimes\partial_w \\ &\qquad + w\otimes\partial_w\otimes v\otimes\partial_v + w\otimes\partial_w\otimes w\otimes\partial_w)] \\ &= -\wedge(v\otimes v\otimes\partial_v\otimes\partial_v + w\otimes v\otimes\partial_v\otimes\partial_w + v\otimes w\otimes\partial_w\otimes\partial_v \\ &\qquad + w\otimes w\otimes\partial_w\otimes\partial_w + \delta\otimes v\otimes\partial_v + \delta\otimes w\otimes\partial_w) \\ &= -\wedge(w\otimes v\otimes\partial_v\otimes\partial_w + v\otimes w\otimes\partial_w\otimes\partial_v + \delta\otimes E). \end{aligned} \tag{4.11}$$

Equation (4.11) gives the formula:

$$\begin{aligned} \wedge(E\otimes E + \delta\otimes E) &= -\wedge(w\otimes v\otimes\partial_v\otimes\partial_w + v\otimes w\otimes\partial_w\otimes\partial_v) \\ &= 2(v\wedge w)\otimes(\partial_v\wedge\partial_w). \end{aligned} \tag{4.12}$$

Since $v\wedge w \neq 0$, on the space Δ, the field equation is reduced just to the equation $(\partial_v\wedge\partial_w)(\psi) = 0$.

Alternatively, we may pull back the function ψ to the space H, along the maps $X_\pm$, using the relations $X = \pm 2\zeta^+\wedge\zeta^-$, for $\zeta \in H$. This leads instead to the formula:

$$\begin{aligned} \wedge(E\otimes E + \delta\otimes E) &= -\wedge(\overline{\zeta}\otimes\zeta\otimes\partial_\zeta\otimes\partial_{\overline{\zeta}} + \zeta\otimes\overline{\zeta}\otimes\partial_{\overline{\zeta}}\otimes\partial_\zeta) \\ &= 2(\zeta\wedge\overline{\zeta})\otimes(\partial_\zeta\wedge\partial_{\overline{\zeta}}). \end{aligned} \tag{4.13}$$

Since $\zeta\wedge\overline{\zeta} \neq 0$, the field equation is reduced just to the equation $(\partial_\zeta\wedge\partial_{\overline{\zeta}})(\psi) = 0$. This equation in turn may be reformulated by introducing the $(0,1)$-form on the space H, $\beta(\psi) \equiv \iota(\partial_\zeta\psi)(\mathrm{d}\overline{\zeta})$.

Lemma 4.9. *The* $(0,1)$ *form* $\beta(\psi)$ *on* H *is* $\overline{\partial}$*-closed,* $\overline{\partial}\beta(\psi) = 0$*, if and only if* ψ *obeys the field equation* $(\partial_\zeta \wedge \partial_{\overline{\zeta}})(\psi) = 0$.

Proof. We have

$$\begin{aligned}\overline{\partial}\beta(\psi) &= -\iota((\partial_{\overline{\zeta}}[\iota(\partial_\zeta\psi))\,\mathrm{d}\overline{\zeta}])\,\mathrm{d}\overline{\zeta} \\ &= ([\iota(\iota(\partial_{\overline{\zeta}} \wedge \partial_\zeta(\psi))\,\mathrm{d}\overline{\zeta}]))\,\mathrm{d}\overline{\zeta} = 0,\end{aligned}$$

if and only if $(\partial_\zeta \wedge \partial_{\overline{\zeta}})(\psi) = 0$. ∎

5. The invariant Hilbert transform

Let X be a two-dimensional oriented real vector space. Then the Hilbert transform is a complex structure for the space $C^\infty(X,-1)$, i.e. a linear isomorphism $H : C^\infty(X,-1) \to C^\infty(X,-1)$, such that $-H^2$ is the identity operator. For the case that X is $\mathbb{R}^2$, with its standard orientation and for any given $f \in C^\infty(\mathbb{R}^2,-1)$, the traditional formula for the transform $H(f)$ will be described next. First define an auxiliary function $h(f)(\theta,x,y)$ for $(\theta,x,y) \in \mathbb{R}\times(\mathbb{R}^2)'$ by the formula, valid whenever $x\sin\theta - y\cos\theta \neq 0$:

$$(x\sin\theta - y\cos\theta)h(f)(\theta,x,y) = f(\cos\theta,\sin\theta) - (x\cos\theta + y\sin\theta)f(x,y). \tag{5.1}$$

Lemma 5.1. *The function* $h(f)$ *of equation (5.1) has a unique smooth extension (still called* $h(f)$*) to the space* $\mathbb{R}\times(\mathbb{R}^2)'$.

Proof. The differential of the function $x\sin\theta - y\cos\theta$ is

$$(\sin\theta)\,\mathrm{d}x - (\cos\theta)\,\mathrm{d}y + (x\cos\theta + y\sin\theta)\,\mathrm{d}\theta,$$

which never vanishes. So we need only to show that the right-hand side of equation (5.1) vanishes whenever the quantity $x\sin\theta - y\cos\theta$ vanishes for some $(\theta,x,y) \in \mathbb{R}\times(\mathbb{R}^2)'$. But if $(\theta,x,y) \in \mathbb{R}\times(\mathbb{R}^2)'$ and $x\sin\theta - y\cos\theta = 0$, we have $y = t\sin\theta$ and $x = t\cos\theta$, for some (unique) $0 \neq t \in \mathbb{R}$. The right-hand side of equation (5.1) may then be rewritten as $f(\cos\theta,\sin\theta) - (t\cos^2\theta + t\sin^2\theta)f(t\cos\theta,t\sin\theta) = 0$, as required, using the homogeneity of f. ∎

Note that the function $h(f)$ is periodic: $h(f)(\theta+\pi,x,y) = h(f)(\theta,x,y)$, for all $(\theta,x,y) \in \mathbb{R}\times(\mathbb{R}^2)'$. The Hilbert transform of $f \in C^\infty(\mathbb{R}^2,-1)$, $H(f) \in C^\infty(\mathbb{R}^2,-1)$ is defined by the formula:

$$H(f)(x,y) \equiv (2\pi)^{-1}\int_{-\pi}^{\pi} h(f)(\theta,x,y)\,\mathrm{d}\theta = \pi^{-1}\int_{-\pi/2}^{\pi/2} h(f)(\theta,x,y)\,\mathrm{d}\theta. \tag{5.2}$$

Equation (5.2) is valid for all $(x,y) \in (\mathbb{R}^2)'$. We may verify that this expression agrees with more traditional formulas for the Hilbert transform as follows.

Put $x = r\cos\alpha$ and $y = r\sin\alpha$, for $0 < r \in \mathbb{R}$ and $\alpha \in \mathbb{R}$. Then we have, successively,

$$\begin{aligned} H(f)(x,y) &= \pi^{-1}\int_{-\pi/2}^{\pi/2} h(f)(\theta+\alpha,x,y)\,\mathrm{d}\theta \\ &= \text{p.v.}(\pi r)^{-1}\int_{-\pi/2}^{\pi/2}[\cos\alpha\sin(\theta+\alpha)-\sin\alpha\cos(\theta+\alpha)]^{-1} \\ &\qquad\times[f(\cos(\theta+\alpha),\sin(\theta+\alpha)) \\ &\qquad-(\cos\alpha\cos(\theta+\alpha)+\sin\alpha\sin(\theta+\alpha))f(\cos\alpha,\sin\alpha)]\,\mathrm{d}\theta \\ &= \text{p.v.}(\pi r)^{-1}\int_{-\pi/2}^{\pi/2}(\sin\theta)^{-1} \\ &\qquad\times[f(\cos(\theta+\alpha),\sin(\theta+\alpha))-\cos\theta f(\cos\alpha,\sin\alpha)]\,\mathrm{d}\theta \\ &= \text{p.v.}(\pi r)^{-1}\int_{-\pi/2}^{\pi/2}(\sin\theta)^{-1}f(\cos(\theta+\alpha),\sin(\theta+\alpha))\,\mathrm{d}\theta \\ &= \text{p.v.}(\pi r)^{-1}\int_{\alpha-\pi/2}^{\alpha+\pi/2}(\sin(\theta-\alpha))^{-1}f(\cos\theta,\sin\theta)\,\mathrm{d}\theta \\ &= \text{p.v.}(\pi)^{-1}\int_{-\infty}^{\infty}(xt-y)^{-1}f(1,t)\,\mathrm{d}t \\ &= \text{p.v.}(\pi)^{-1}\int_{-\infty}^{\infty}(x-yu)^{-1}f(u,1)\,\mathrm{d}u. \end{aligned} \tag{5.3}$$

In equation (5.3), the term p.v. stands for the Cauchy principal value of the integral. It is not needed for the first line, since the integrand $h(f)$ is everywhere smooth. In going from the third line to the fourth, we have dropped the integral of $\cot\theta$. This is valid since $\cot\theta$ is an odd function, only singular at the origin in the range $-\pi/2 \leqslant \theta \leqslant \pi/2$. So the Cauchy principal value of its integral is zero. The final line of equation (5.3) gives two traditional formulas for the Hilbert transform. The first is valid provided $x \neq 0$ and the second is valid provided $y \neq 0$.

Given any $f \in C^\infty(\mathbb{R}^2,-1)$, we also construct a pair of holomorphic functions, $H^\pm(f)$, as follows:

$$\begin{aligned} H^\pm(f)(x,y) &\equiv (2\mathrm{i}\pi)^{-1}\int_{-\pi/2}^{\pi/2} f(\cos\theta,\sin\theta)(x\sin\theta-y\cos\theta)^{-1}\,\mathrm{d}\theta \\ &= (2\mathrm{i}\pi)^{-1}\int_{-\infty}^{\infty} f(1,t)(xt-y)^{-1}\,\mathrm{d}t = (2\mathrm{i}\pi)^{-1}\int_{-\infty}^{\infty} f(u,1)(x-yu)^{-1}\,\mathrm{d}u. \end{aligned} \tag{5.4}$$

Here x and y are complex and non-zero; the domain of $H^+(f)$ is given by $\mathrm{Im}(y/x) > 0$, and the domain of $H^-(f)$ is given by $\mathrm{Im}(y/x) < 0$.

Note that the functions $H^\pm(f)$ are each homogeneous of degree minus one in the pair (x,y). Then the fundamental properties of the Hilbert transform may be summarized as follows (Muskhelishvili 1968; Bell 1992)). First, the functions $H^\pm(f)$ possess smooth extensions (still denoted $H^\pm(f)$) to the closure of their respective domains in $(\mathbb{C}^2)'$. Denote by ${}_{\mathbb{R}}H^\pm(f)$ the restriction of (the extensions of) $H^\pm(f)$ to

$(\mathbb{R}^2)'$. Then the functions ${}_{\mathbb{R}}H^{\pm}(f)$ are elements of $C^\infty(\mathbb{R}^2, -1) \otimes \mathbb{C}$ and we have the relations,

$$ {}_{\mathbb{R}}H^{+}(f) - {}_{\mathbb{R}}H^{-}(f) = f, \qquad {}_{\mathbb{R}}H^{+}(f) + {}_{\mathbb{R}}H^{-}(f) = -\mathrm{i}H(f). \tag{5.5}$$

Further, the functions $H^{\pm}(f)$, satisfying these properties, are unique. Note that it follows immediately from the uniqueness that the operator $-H^2$ is the identity operator. Also we have the complex conjugation relations,

$$[H^{+}(f)(x,y)]' = -H^{-}(f)(x',y'), \tag{5.6}$$

valid for all $(x,y) \in (\mathbb{C}^2)'$, with $x \neq 0$ and $\mathrm{Im}(y/x) > 0$;

$$[{}_{\mathbb{R}}H^{+}(f)(x,y)]' = -{}_{\mathbb{R}}H^{-}(f)(x,y), \tag{5.7}$$

valid for all $(x,y) \in (\mathbb{R}^2)'$.

Example 5.2. Consider the case that $f(x,y) \equiv (\alpha x + \beta y)(ax^2 + 2bxy + cy^2)^{-1}$, for all $(x,y) \in (\mathbb{R}^2)'$, where the real constants α, β, a, b and c satisfy the conditions: α and β are not both zero, $0 < a$, $0 < c$ and $0 < ac - b^2$. Define the quantities $k \equiv (ac-b^2)^{1/2}$, $\gamma \equiv k^{-1}(a\beta - b\alpha)$, $\delta \equiv k^{-1}(b\beta - c\alpha)$ and $\epsilon \equiv k^{-2}(c\alpha^2 - 2b\alpha\beta + a\beta^2)$. Note that $\epsilon > 0$. Then we find the following formulas for the functions $H(f)$ and $H^{\pm}(f)$:

$$H(f)(x,y) = (\gamma x + \delta y)(ax^2 + 2bxy + cy^2)^{-1}, \tag{5.8}$$

valid for all $(x,y) \in (\mathbb{R}^2)'$;

$$2H^{+}(f)(x,y) = \epsilon[(\alpha + i\gamma)x + (\beta + \mathrm{i}\delta)y]^{-1}, \tag{5.9}$$

valid for all $(x,y) \in (\mathbb{C}^2)'$, with $x \neq 0$ and $\mathrm{Im}(y/x) > 0$;

$$2H^{-}(f)(x,y) = \epsilon[(\mathrm{i}\gamma - \alpha)x + (\mathrm{i}\delta - \beta)y)]^{-1}, \tag{5.10}$$

valid for all $(x,y) \in (\mathbb{C}^2)'$, with $x \neq 0$ and $\mathrm{Im}(y/x) < 0$.

Note that $\mathrm{Im}[(\alpha+\mathrm{i}\gamma)/(\beta+\mathrm{i}\delta)] = (\beta\gamma - \alpha\delta)(\beta^2+\delta^2)^{-1} > 0$, since $\beta\gamma - \alpha\delta = k\epsilon > 0$, so both the functions $H^{\pm}(f)$ are well defined on their domains.

We next reformulate the Hilbert transform invariantly. So let X be any oriented two-dimensional real vector space and let $f \in C^\infty(X, -1)$ be given.

Lemma 5.3. *There exists a unique smooth function $h(f) \in C^\infty(X' \times X')$, homogeneous of degrees minus one in each of its X' arguments and taking values in X^*, such that we have the following relation, valid for all $(u,v) \in X' \times X'$:*

$$f(u)u - f(v)v = \iota(h(f)(u,v))(u \wedge v). \tag{5.11}$$

An equivalent definition of $h(f)(u,v)$ is

$$\iota(h(f)(u,v))(u) = -f(v), \qquad \iota(h(f)(u,v))(v) = -f(u), \tag{5.12}$$

valid for all $(u,v) \in X' \times X'$.

Proof. If $u \in X$ and $v \in X$ are such that $u \wedge v \neq 0$, then the pair u, v forms a basis of X. Let u^*, v^* be the dual basis of X^* (so we have $u^*(u) = v^*(v) = 1$ and $v^*(u) = u^*(v) = 0$). Then put $h(f)(u,v) \equiv -f(v)u^* - f(u)v^*$. As the vectors u and v vary, this gives a well-defined function $h(f)$ satisfying the requirements of the lemma whenever $u \wedge v \neq 0$. It remains to show that the function $h(f)$ extends smoothly to the space of all $(u, v) \in X'xX'$. Fix the vector $u \in X'$. Let the pair u, w be a basis of X for some fixed vector $w \in X$, with dual basis u^*, w^*. If now $v \in X'$, then $v = su + tw$, for some unique real numbers s and t, not both zero. Then we have the following relation, valid for any real $t \neq 0$:

$$h(f)(u, v) = -f(su + tw)u^* - t^{-1}(f(u) - sf(su + tw))w^*. \tag{5.13}$$

By Taylor's theorem there exists a unique smooth function $k(f)(s, t, u, w)$, defined for all $(s, t) \in (\mathbb{R}^2)'$ and all pairs $u, w \in X \times X$, with $u \wedge w \neq 0$, such that whenever $t \neq 0$ and $u \wedge w \neq 0$, we have the relation:

$$t^{-1}(f(u) - sf(su + tw)) = k(f)(s, t, u, w). \tag{5.14}$$

Then the formula $h(f)(u, v) \equiv -f(su + tw)u^* - k(f)(s, t, u, w)w^*$, with $v = su + tw$ extends the function $h(f)$ smoothly (and uniquely, by continuity) to the space of all $(u, v) \in X' \times X'$. ∎

Note that by uniqueness, we have $h(f)(u, v) = h(f)(v, u)$, for all $(u, v) \in X' \times X'$.

Next consider the one form $\eta(f)(v) \equiv \iota(h(f)(u, v))(\mathrm{d}u)$. Then $\eta(f)(v)$ is a smooth one form, globally defined on X', smoothly varying as $v \in X'$ varies and homogeneous of degree minus one in the variable $v \in X'$.

Lemma 5.4. *The one-form $\eta(f)(v)$ is closed: $\mathrm{d}\eta(f)(v) = 0$, for all $v \in X'$.*

Proof. By continuity, we may assume $u \wedge v \neq 0$. Using lowercase Latin abstract indices for the tensor algebra of X, we have $\eta(f)(v) = h(f)_a(u, v)\, \mathrm{d}u^a$, where $h(f)_a(u, v)$ obeys the defining relations $u^a h(f)_a(u, v) = -f(v)$ and $v^a h(f)_a(u, v) = -f(u)$, given in equation (5.12). If ∂_a denotes the derivative with respect to the variable u^a, it must be shown that $\partial_a h(f)_b$ is symmetric in its index pair. So, since $u \wedge v \neq 0$, it is sufficient to demonstrate that $0 = (u^a v^b - v^a u^b)\partial_a h(f)_b$. Now, using the defining relations of $h(f)_a$ and the formula $\partial_a u^b = \delta_a^b$, we have

$$\begin{aligned}(u^a v^b - v^a u^b)\partial_a h(f)_b &= u^a \partial_a (v^b h(f)_b) - v^a \partial_a (u^b h(f)_b) + v^a \delta_a^b h(f)_b \\ &= v^a h(f)_a - u^a \partial_a (f(u)) + v^a \partial_a (f(v)) \\ &= -f(u) - u^a \partial_a (f(u)) = 0,\end{aligned}$$

as required. ∎

Corollary 5.5. *For each $v \in X'$, the form $\eta(f)(v)$ defines an element $[\eta(f)(v)]$ of $H^1(X', R)$, depending smoothly on the vector v. There is then a unique function $H_X(f) \in C^\infty(X, -1)$, such that $H_X(f)(v) = (2\pi)^{-1} \int_X([\eta(f)(v)])$, for all $v \in X'$.*

Definition 5.6. For any X a real two-dimensional oriented vector space and for any $f \in C^\infty(X, -1)$, the function $H_X(f)$ of corollary 5.5 is the Hilbert transform of f. The Hilbert transform for the space X is the linear map, $H_X : C^\infty(X, -1) \to C^\infty(X, -1)$, which takes any function $f \in C^\infty(X, -1)$ to its Hilbert transform $H_X(f)$.

Lemma 5.7. *$H_Y(f\lambda) = H_X(f)\lambda$, for any $f \in C^\infty(X, -1)$ and for any $\lambda : Y \to X$, an isomorphism of oriented two-dimensional real vector spaces X and Y.*

Proof. The construction of the Hilbert transform is obviously functorial, with respect to such isomorphisms. ∎ OK?

Lemma 5.8. *If $X = \mathbb{R}^2$, with its standard orientation, then the operator H_X agrees with the operator H defined in equation (5.2) above.*

Proof. Take $X = \mathbb{R}^2$ and map the real interval $[-\pi, \pi]$ to $\mathbb{R}^2$ by the formula: $s(\theta) = (\cos\theta, \sin\theta)$, for any $\theta \in [-\pi, \pi]$. Then we have the integral formula for the quantity $H_X(f)(x, y)$, for any $(x, y) \in (\mathbb{R}^2)'$:

$$H_X(f)(x,y) = (2\pi)^{-1} \int_{-\pi}^{\pi} s^*(\eta(f)(x,y))\,\mathrm{d}\theta. \tag{5.15}$$

Put $u = (p, q) \in (\mathbb{R}^2)'$ and $v = (x, y) \in (\mathbb{R}^2)'$. Then $\eta(f)(x, y) = h_0(f)\,\mathrm{d}p + h_1(f)\,\mathrm{d}q$, where the functions $h_0(f)(p, q, x, y)$ and $h_1(f)(p, q, x, y)$ are given, whenever $qx - py \neq 0$ by the formulas (from equation (5.11)):

$$\left.\begin{aligned} (qx - py)h_0(f)(p,q,x,y) &= yf(x,y) - qf(p,q), \\ (qx - py)h_1(f)(p,q,x,y) &= -xf(x,y) + pf(p,q). \end{aligned}\right\} \tag{5.16}$$

Then we have the pullback formula:

$$\begin{aligned} s^*(\eta(f)(x,y)) &= [-h_0(f)(\cos\theta, \sin\theta, x, y)\sin\theta + h_1(f)(\cos\theta, \sin\theta, x, y)\cos\theta]\,\mathrm{d}\theta \\ &= g(f)(\theta, x, y)\,\mathrm{d}\theta, \end{aligned}$$

where the smooth function $g(f)(\theta, x, y)$ is defined for all $(\theta, x, y) \in \mathbb{R} \times (\mathbb{R}^2)'$, by the formula:

$$\begin{aligned} (x\sin\theta - y\cos\theta)g(f)(\theta,x,y) &= -\sin\theta(x\sin\theta - y\cos\theta)h_0(f)(\cos\theta,\sin\theta,x,y) \\ &\quad + \cos\theta(x\sin\theta - y\cos\theta)h_1(f)(\cos\theta,\sin\theta,x,y) \\ &= -\sin\theta[yf(x,y) - f(\cos\theta,\sin\theta)\sin\theta] \\ &\quad + \cos\theta[-xf(x,y) + f(\cos\theta,\sin\theta)\cos\theta] \\ &= f(\cos\theta,\sin\theta) - (x\cos\theta + y\sin\theta)f(x,y). \end{aligned} \tag{5.17}$$

Equation (5.17) is valid whenever $x\sin\theta - y\cos\theta \neq 0$ and elsewhere the function $g(f)$ extends by continuity.

So now we have the expression:

$$H_X(f)(x,y) = (2\pi)^{-1} \int_{-\pi}^{\pi} g(f)(\theta,x,y)\,\mathrm{d}\theta. \tag{5.18}$$

But equations (5.17) and (5.18) are in exact agreement with equations (5.1) and (5.2), so the lemma holds. ■

Corollary 5.9. *The operator $-H_X^2$ is the identity operator, for any X.*

Proof. Pick an isomorphism $\lambda : X \to \mathbb{R}^2$ and apply lemmas 5.7 and 5.8. ■

Finally, we provide an invariant definition of the functions $H^{\pm}(f)$. Denote by $D(X)$ the domain of all $\zeta \in X_{\mathbb{C}}$, such that $\zeta \wedge \overline{\zeta} \neq 0$. Then define a complex-valued one form $\rho(f)(\zeta)$ on the space X' by the formula, for any $\zeta \in D(X)$:

$$(u \wedge z)(\rho(f)(\zeta))(u) = \tfrac{1}{2}\mathrm{i}f(u)u \wedge \mathrm{d}u. \tag{5.19}$$

If $u \in X$ and $\zeta \in D(X)$, denote by (u^*, ζ^*) the basis dual to the basis (ζ, u) of $X_{\mathbb{C}}$. Then $\iota(u^*)(u \wedge \zeta) = -u$ and $\iota(u^*)(u \wedge \mathrm{d}u) = -u(\iota(u^*)(\mathrm{d}u))$ so we may also write out the one form $\rho(f)(\zeta)$ explicitly as follows:

$$(\rho(f)(\zeta))(u) = \tfrac{1}{2}\mathrm{i}f(u)(\iota(u^*)(\mathrm{d}u)). \tag{5.20}$$

Lemma 5.10. *The one form $\rho(f)(\zeta)$ is closed:* $\mathrm{d}\rho(f)(\zeta)=0$, *for any $\zeta\in D(X)$.*

Proof. Since $\iota(u^*)(\zeta)=1$, we have $\iota(\mathrm{d}u^*)(\zeta)=0$. It follows that $du^*=\alpha\zeta^*$, for some one form α. Since $\iota(u^*)(\zeta)=0$, we have $\iota(\mathrm{d}u^*)(u)+\iota(u^*)(\mathrm{d}u)=0$, so $\alpha\iota(\zeta^*)(u)+\iota(u^*)(\mathrm{d}u)=0$, so $\alpha=-\iota(u^*)(\mathrm{d}u)$. Then we have, from equation (5.20):

$$\begin{aligned}2\mathrm{i}[\mathrm{d}(\rho(f)(\zeta))](u)&=\iota(\mathrm{d}u^*)(u)df-f(u)\iota(\mathrm{d}u^*)(\mathrm{d}u)\\&=(\iota(u^*)(\mathrm{d}u))(\iota(\partial_u f)(\mathrm{d}u))+f(u)(\iota(u^*)(\mathrm{d}u)\iota(\zeta^*)(\mathrm{d}u)).\end{aligned}\tag{5.21}$$

Now let $(\partial_u f)(u)=p(u)u^*+q(u)\zeta^*$, for all $u\in X'$. Then in particular we have $\iota(\partial_u f)(u)=-f(u)=q(u)$. Inserting these relations into equation (5.21), we get

$$\begin{aligned}-2\mathrm{i}[\mathrm{d}(\rho(f)(\zeta))](u)&=p(u)(\iota(u^*)(\mathrm{d}u))(\iota(u^*)(\mathrm{d}u))-f(u)(\iota(\zeta^*)(\mathrm{d}u))(\iota(u^*)(\mathrm{d}u))\\&\quad-f(u)(\iota(u^*)(\mathrm{d}u))(\iota(\zeta^*)(\mathrm{d}u))=0.\end{aligned}\tag{5.22}$$

■

Corollary 5.11. *For each fixed $\zeta\in D(X)$, the form $\rho(f)(\zeta)$ defines an element $[\rho(f)(\zeta)]$ of $H^1(X',C)$.*

Denote by $P(f)(\zeta)\in\mathbb{C}$ the quantity $(2\pi)^{-1}\int_X[\rho(f)(z)]$ and by $P(f)$ the function on $D(X)$ whose value at any $\zeta\in D(X)$ is $P(f)(\zeta)$.

Corollary 5.12. *As $\zeta\in D(X)$ varies, keeping f fixed, the function $P(f)$ is holomorphic on $D(X)$ and homogeneous of degree -1 in the variable ζ.*

Now the domain $D(X)$ is the disjoint union of two open subsets: $D(X)=D_+(X)\cup D_-(X)$, where $D_\pm(X)$ is by definition the set of all $\zeta\in D(X)$, such that $(\zeta^+,\pm\zeta^-)$ is an oriented basis of X.

Definition 5.13. The functions $H^\pm_X(f)$ are by definition the restrictions of the function $P(f)$ to the domains $D_\pm(X)$, respectively.

The functions $H^\pm_X(f)$ of definition 5.13 are each holomorphic and homogeneous of degree minus one.

Lemma 5.14. *We have $H^\pm_Y(f\lambda)=H^\pm_X(f)\lambda_{\mathbb{C}}$, for any $f\in C^\infty(X,-1)$ and for any $\lambda:Y\to X$, an isomorphism of oriented two-dimensional real vector spaces X and Y, with complexification $\lambda_{\mathbb{C}}:Y_{\mathbb{C}}\to X_{\mathbb{C}}$.*

Proof. The construction of the quantity $H^\pm_X(f)$ is obviously functorial, with respect to such isomorphisms. ■

Theorem 5.15. *The functions $H^\pm_X(f)$ extend smoothly to functions defined on the closure of the domains $D_\pm(X)$ in $(X_{\mathbb{C}})'$. Denote by ${}_{\mathbb{R}}H^\pm_X(f)$ the restrictions to X' of the extensions of $H^\pm_X(f)$. Then ${}_{\mathbb{R}}H^\pm_X(f)\in C^\infty(X,-1)\otimes\mathbb{C}$ and we have the decompositions:*

$${}_{\mathbb{R}}H^+_X(f)-{}_{\mathbb{R}}H^-_X(f)=f;\tag{5.23}$$

$${}_{\mathbb{R}}H^+_X(f)+{}_{\mathbb{R}}H^-_X(f)=-\mathrm{i}H_X(f).\tag{5.24}$$

Furthermore, these decompositions are unique.

Proof. Using a suitable isomorphism $\lambda : X \to \mathbb{R}^2$, by lemma 5.14, it is sufficient to establish this result when $X = \mathbb{R}^2$, equipped with its standard orientation. Henceforth assume $X = \mathbb{R}^2$, with its standard orientation. Then we must show that $H_X^{\pm}(f) = H^{\pm}(f)$, for any $f \in C^{\infty}(X, -1)$, where $H^{\pm}(f)$ is defined by equation (5.4).

Let $(x, y) \in \mathbb{C}^2$ be such that $x\overline{y} - y\overline{x} \neq 0$. Note that neither x nor y is zero. Now the ordered pair of vectors $\mathrm{Re}(x, y)$ and $\mathrm{Im}(x, y)$ form an oriented basis of $\mathbb{R}^2$, with its usual orientation, if and only if

$$\begin{aligned} 0 < \mathrm{Re}(x)\,\mathrm{Im}(y) - \mathrm{Im}(x)\,\mathrm{Re}(y) &= (4\mathrm{i})^{-1}[(x+\overline{x})(y-\overline{y}) - (x-\overline{x})(y+\overline{y})] \\ &= (2\mathrm{i})^{-1}[-x\overline{y} + y\overline{x}] = (2\mathrm{i}x\overline{x})^{-1}(y/x - \overline{y}/\overline{x}) \\ &= (x\overline{x})^{-1}\,\mathrm{Im}(y/x). \end{aligned}$$

So $D_{\pm}(\mathbb{R}^2)$ is the domain of all $(x, y) \in \mathbb{C}^2$, such that $x \neq 0$ and $\pm\,\mathrm{Im}(y/x) > 0$. Then we have, using the map s introduced in lemma 5.8 above:

$$H_X^{\pm}(f)(x, y) = (2\pi)^{-1} \int_{-\pi}^{\pi} s^*(\rho(f)(x, y))(\theta)\,\mathrm{d}\theta. \tag{5.25}$$

Now $s^*(\mathrm{d}u)(\theta) = (-\sin\theta, \cos\theta)\,\mathrm{d}\theta$ and when $u = (\cos\theta, \sin\theta)$ we find, from the definition of u^*, given after equation (5.19), that $u^* = (-\sin\theta, \cos\theta)\ (-x\sin\theta + y\cos\theta)^{-1}$. So $s^*(\iota(u^*)(\mathrm{d}u))(\theta) = (-x\sin\theta + y\cos\theta)^{-1}\,\mathrm{d}\theta$. So by equation (5.20), we have

$$s^*(\rho(f)(x, y))(\theta) = -\tfrac{1}{2}\mathrm{i} f(\cos\theta, \sin\theta)(x\sin\theta - y\cos\theta)^{-1}\,\mathrm{d}\theta.$$

Then equation (5.25) becomes the following:

$$\begin{aligned} H_X^{\pm}(f)(x, y) &= (4\mathrm{i}\pi)^{-1} \int_{-\pi}^{\pi} f(\cos\theta, \sin\theta)(x\sin\theta - y\cos\theta)^{-1}\,\mathrm{d}\theta \\ &= (2\mathrm{i}\pi)^{-1} \int_{-\pi/2}^{\pi/2} f(\cos\theta, \sin\theta)(x\sin\theta - y\cos\theta)^{-1}\,\mathrm{d}\theta. \end{aligned} \tag{5.26}$$

Here $H_X^{\pm}(f)(x, y)$ is defined for $\pm\,\mathrm{Im}(y/x) > 0$. Comparing equations (5.4) and (5.26), we find exact agreement. ∎

Corollary 5.16. *For all $\zeta \in (X_{\mathbb{C}})'$ in the closure of $D_+(X)$ and for all $f \in C^{\infty}(X, -1)$, we have the relation,*

$$\overline{(H^+(f)(\zeta))} = -H^-(f)(\overline{\zeta}). \tag{5.27}$$

For all $v \in X'$ and for all $f \in C^{\infty}(X, -1)$, we have the relation,

$$\overline{(H^+(f)(v))} = -H^-(f)(v). \tag{5.28}$$

Proof. These relations follow from the uniqueness of the decomposition of equations (5.23) and (5.24), by taking the complex conjugates of equations (5.23) and (5.24) and using the fact that the function f is real valued. ∎

Example 5.17. Take $f(v) \equiv g(a, v)g(v, v)^{-1}$, for any $v \in X'$, where $a \in X'$ is fixed and g is a symmetric bilinear form on the complex vector space $X_{\mathbb{C}}$, which is real and definite when restricted to the space X. Let $b \in X'$, be the unique vector such that such that $g(a, b) = 0$, $g(b, b) = g(a, a)$ and the pair (a, b) is an oriented basis for X.

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Then we have the following expression for the function $h(f)$, valid for any $(u, v) \in X' \times X'$:

$$h(f)(u, v) = [g(a,)g(u, v) - g(u,)g(a, v) - g(v,)g(a, u)][g(u, u)g(v, v)]^{-1}. \quad (5.29)$$

Using definition 5.6, we find the following formula for the function $H(f)$, valid for any $v \in X'$:

$$H(f)(v) = -g(b, v)g(v, v)^{-1}. \quad (5.30)$$

Then $D_\pm(X) = \zeta \in X_{\mathbb{C}}; \pm \operatorname{Im}(g(\zeta, b)/g(\zeta, a)) > 0$ and the functions $H^\pm(f)$ are given as follows:

$$\pm 2H^\pm(f)(\zeta) = -g(a, a)[-g(\zeta, a) \pm ig(\zeta, b)]^{-1}, \quad (5.31)$$

valid for all $\zeta \in D_\pm(X)$.

6. Application of a sheaf cohomology exact sequence

For any integer m, denote by $\Theta(m)$ the sheaf of germs of holomorphic sections of the holomorphic line bundle over the space $P_{\mathbb{C}}$ of Chern class m.

In this section we take $d(V) = n + 1$, where $n \geqslant 3$, n integral and prove the following theorem.

Theorem 6.1. *The cohomology group $H^1(P_{\mathbb{C}} - P; \Theta(-2))$ vanishes.*

We begin with a standard exact sequence for a triple consisting of X a *compact* complex manifold, S a closed subset of X and L a holomorphic line bundle over X.

For each integer k, $H^k(S, L)$ is by definition the space of all smooth $\overline{\partial}$-closed $(0, k)$ forms with values in L, defined on some open set U containing S, modulo the equivalence relation $\alpha \equiv \beta$, if and only if there exists an open set U' containing S, such that U' is in the domain of definition of both α and β and $\alpha - \beta$, restricted to U' is $\overline{\partial}$-exact.

Lemma 6.2. *There is an exact cohomology sequence:*

$$\cdots \xrightarrow{\epsilon_k} H^k(X, L) \xrightarrow{\rho_k} H^k(S, L) \xrightarrow{\delta_k} H_c^{k+1}(X - S, L) \xrightarrow{\epsilon_{k+1}} H^{k+1}(X, L) \xrightarrow{\rho_{k+1}} \cdots .$$

Here the subscript c refers to cohomology with compact supports.

Proof. The maps of the exact sequence are as follows:

$\rho_k : H^k(X, L) \to H^k(S, L)$ is the natural restriction map.

$\epsilon_k : H_c^{k+1}(X - S, L) \to H^{k+1}(X, L)$ is the natural extension map, extending by zero any form of compact support in $X - S$ to the whole of X.

$\delta_k : H^k(S, L) \to H_c^{k+1}(X - S, L)$ is defined as follows: let a representative $\overline{\partial}$-closed $(0, k)$ form α be given defined on an open subset U containing S. After shrinking the domain of α and smoothing to zero, we can construct a $(0, k)$ form β, globally defined on X, which agrees with α on an open subset U' of U, where U' contains S. Put $\gamma \equiv \overline{\partial}\beta$. Then it is clear that γ has compact support in $X - S$ and $\overline{\partial}\gamma = 0$, so γ represents an element of $H_c^{k+1}(X - S, L)$. By definition this element is the image under δ_k of the element of $H^k(S, L)$ represented by α.

It is easy to check that the maps are well defined and that the sequence is indeed exact. ∎

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Corollary 6.3. *There is an exact cohomology sequence, for any integer m:*

$$H^k(P_{\mathbb{C}}, \Theta(m)) \xrightarrow{\rho_k} H^k(P, \Theta(m)) \xrightarrow{\delta_k} H_c^{k+1}(P_{\mathbb{C}} - P, \Theta(m)) \xrightarrow{\epsilon_{k+1}} H^{k+1}(P_{\mathbb{C}}, \Theta(m)).$$

Lemma 6.4. *The group $H^1(P_{\mathbb{C}}, \Theta(-2))$ vanishes.*

Proof. A direct proof that the group $H^1(P_{\mathbb{C}}, \Theta(-2))$ vanishes is given in corollary 7.15 below. Alternatively, this lemma is a special case of the Borel–Weil–Bott theorem, which in particular demonstrates that the sheaf cohomology of holomorphic line bundles over complex projective space is concentrated in the lowest and highest dimensions. ∎

Corollary 6.5. *The group $H^{n-1}(P_{\mathbb{C}}, \Theta(1-n))$ vanishes.*

Proof. By Serre duality, the group $H^{n-1}(P_{\mathbb{C}}, \Theta(1-n))$ is dual to the group $H^{n,1}(P_{\mathbb{C}}, \Theta(n-1))$. In turn the group $H^{n,1}(P_{\mathbb{C}}, \Theta(n-1))$ is naturally isomorphic to the group $H^1(P_{\mathbb{C}}, \Theta(-2)) \otimes \Omega^{n+1}(V_{\mathbb{C}}^*)$. Now, by lemma 6.4, the group $H^1(P_{\mathbb{C}}, \Theta(-2))$ vanishes. Therefore, the group $H^{n,1}(P_{\mathbb{C}}, \Theta(n-1))$ vanishes. So by Serre duality again, the group $H^{n-1}(P_{\mathbb{C}}, \Theta(1-n))$ must vanish also, as required. (Of course this result also is included in the Borel–Weil–Bott theorem.) ∎

Lemma 6.6. *The group $H^{n-2}(P, \Theta(1-n))$ vanishes.*

Proof. If U is open and U contains P, then there is an open *Stein* submanifold U' of $P_{\mathbb{C}}$, such that $U \supset U' \supset P$. So the group $H^{n-2}(U', \Theta(1-n))$ vanishes, since $n \geqslant 3$. So the group $H^{n-2}(P, \Theta(1-n))$ vanishes. ∎

Proof of theorem 6.1. Consider the part of the sequence of corollary 6.3, with $m = 1 - n$ starting at the group $H^{n-2}(P, \Theta(1-n))$:

$$H^{n-2}(P, \Theta(1-n)) \xrightarrow{\delta_{n-2}} H_c^{n-1}(P_{\mathbb{C}} - P, \Theta(1-n)) \xrightarrow{\epsilon_{n-1}} H^{n-1}(P_{\mathbb{C}}, \Theta(1-n)).$$

By lemma 6.6, the group $H^{n-2}(P, \Theta(1-n))$ vanishes. By corollary 6.5, the group $H^{n-1}(P_{\mathbb{C}}, \Theta(1-n))$ vanishes. Therefore, this part of the exact sequence becomes: $0 \to H_c^{n-1}(P_{\mathbb{C}} - P, \Theta(1-n)) \to 0$. Since the sequence is exact, we deduce that the group $H_c^{n-1}(P_{\mathbb{C}} - P, \Theta(1-n))$ vanishes. By Serre duality the group $H_c^{n-1}(P_{\mathbb{C}} - P, \Theta(1-n))$ is dual to the group $H^{n,1}(P_{\mathbb{C}} - P; \Theta(n-1))$. It follows that the group $H^{n,1}(P_{\mathbb{C}} - P; \Theta(n-1))$ also vanishes. The group $H^{n,1}(P_{\mathbb{C}} - P; \Theta(n-1))$ in turn is isomorphic to the group $H^1(P_{\mathbb{C}} - P; \Theta(-2)) \otimes \Omega^{n+1}(V_{\mathbb{C}}^*)$. Therefore, the group $H^1(P_{\mathbb{C}} - P; \Theta(-2)) \otimes \Omega^{n+1}(V_{\mathbb{C}}^*)$ also vanishes. But the space $\Omega^{n+1}(V_{\mathbb{C}}^*)$ is a one-dimensional vector space, so is non-zero, so the group $H^1(P_{\mathbb{C}} - P; \Theta(-2))$ must vanish, as required. ∎

Corollary 6.7. *If β is a $\overline{\partial}$-closed $(0,1)$ form defined globally on $P_{\mathbb{C}} - P$, taking values in $\Theta(-2)$, then there exists a unique smooth section α, of $\Theta(-2)$, globally defined on $P_{\mathbb{C}} - P$, such that $\overline{\partial}\alpha = \beta$.*

Proof. Since β is $\overline{\partial}$-closed, it represents an element of the cohomology group $H^1(P_{\mathbb{C}} - P; \Theta(-2))$. But this group vanishes by theorem 6.1. Therefore, β is $\overline{\partial}$-exact, so some global (on $P_{\mathbb{C}} - P$) section α of $\Theta(-2)$ exists, such that $\overline{\partial}\alpha = \beta$. If also another global section τ of $\Theta(-2)$ exists, with $\overline{\partial}\tau = \beta$, then we have $\overline{\partial}(\alpha - \tau) = 0$, so the section $\alpha - \tau$ is holomorphic. Let $y \in P_{\mathbb{C}} - P$. Then y lies on at least one (actually

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infinitely many) projective line of $P_{\mathbb{C}}$, which does not intersect the subspace P. The restriction of $\alpha - \tau$ to any such line must be zero, since the sheaf $\Theta(-2)$ has no non-zero global holomorphic sections on the projective line (by Liouville's theorem). Therefore, the functions α and t agree on such a line. In particular, $\alpha(y) = \tau(y)$. Since y was arbitrary in $P_{\mathbb{C}} - P$, this establishes that $\alpha = \tau$ and the corollary is proved. ∎

7. The solution of the problem $\overline{\partial}\alpha = \beta$

By corollary 6.7, if β is a $\overline{\partial}$-closed $(0,1)$ form taking values in $\Theta(-2)$ and globally defined on $P_{\mathbb{C}} - P$, then a unique section α of $\Theta(-2)$ exists, smooth and global on $P_{\mathbb{C}} - P$, such that $\overline{\partial}\alpha = \beta$. One may then ask whether or not a formula can be given for α in terms of β. Remarkably, such a formula does exist and will be given in this section.

For any $(\zeta, \eta) \in V_{\mathbb{C}}^2$, with $\zeta \wedge \eta \neq 0$, denote by $L(\zeta, \eta)$ the projective line in $P_{\mathbb{C}}$, through the points $p_{\mathbb{C}}(\zeta)$ and $p_{\mathbb{C}}(\eta)$. Equip the line $L(\zeta, \eta)$ with its standard orientation. For any open set W in $P_{\mathbb{C}}$, we introduce the following spaces.

1. $H(W)$: the inverse image of W in the space $V'_{\mathbb{C}}$, under the natural projection $p_{\mathbb{C}}$.

2. $L(W)$: the space of all pairs $(\zeta, \eta) \in V_{\mathbb{C}}^2$, with $\zeta \wedge \eta \neq 0$, such that the line $L(\zeta, \eta)$ lies entirely in W.

3. $M(W)$: the space of all triples $(\zeta, \eta, u) \in V_{\mathbb{C}}^2 \times P_{\mathbb{C}}$, such that $(\zeta, \eta) \in L(W)$ and $u \in L(\zeta, \eta)$.

4. $N(W)$: the submanifold of $V_{\mathbb{C}}^3$, consisting of all triples $(\zeta, \eta, \upsilon) \in V_{\mathbb{C}}^2 \times V'_{\mathbb{C}}$, such that $(\zeta, \eta, p_{\mathbb{C}}(\upsilon)) \in M(W)$.

We also introduce the following maps.

1. $w : M(W) \to W$, $w(\zeta, \eta, u) = u$, for any $(\zeta, \eta, u) \in M(W)$.

2. $n : N(W) \to W$, $n(\zeta, \eta, \upsilon) = p_{\mathbb{C}}(\upsilon)$, for any $(\zeta, \eta, \upsilon) \in N(W)$.

3. $N : N(W) \to H(W)$, $N(\zeta, \eta, \upsilon) = \upsilon$, for any $(\zeta, \eta, \upsilon) \in N(W)$.

For any fixed $(\zeta, \eta) \in L(W)$, denote by $M(\zeta, \eta)$ the space of all triples $(\zeta, \eta, u) \in M(W)$, such that $u \in L(\zeta, \eta)$. Then the map w, restricted to the space $M(\zeta, \eta)$ is an oriented diffeomorphism onto its image, the space $L(\zeta, \eta)$.

For W open in $P_{\mathbb{C}}$, W will be said to be admissible if and only if, given any point y of W, the space of all projective lines through y, lying entirely in W, is non-empty and connected. Throughout this section, until corollary 7.19, W will refer to an admissible open subset of $P_{\mathbb{C}}$ and β will be a $\overline{\partial}$-closed (not necessarily $\overline{\partial}$-exact) smooth $(0,1)$ form defined globally on W, taking values in the sheaf $\Theta(-2)$.

If w is any form defined on the space $M(W)$, and if $(\zeta, \eta) \in L(W)$, denote by $w(\zeta, \eta)$ the restriction of w to the space $M(\zeta, \eta)$. If w is a two form on $M(W)$ and takes values in an abelian group A and if $(\zeta, \eta) \in L(W)$, denote by $[w(\zeta, \eta)]$ the class represented by $w(\zeta,\eta)$ in the cohomology group $H^2(M(\zeta,\eta),A)$. Then define $L(w)\in C^\infty(L(W),A)$ by the formula, for any $(\zeta,\eta)\in L(W)$:

$$L(w)(\zeta,\eta)\equiv\int_{M(\zeta,\eta)}[w(\zeta,\eta)]. \tag{7.1}$$

In equation (7.1), $\int_{M(\zeta,\eta)}$ is the canonical isomorphism:

$$\int_{M(\zeta,\eta)}: H^2(M(\zeta,\eta),A)\to A$$

for any abelian group A.

Let the pair (ζ,η) provide coordinate functions for the space $L(W)$, and υ a coordinate for the space $H(W)$, where each of the quantities ζ, η and υ takes values in $V_{\mathbb{C}}$. Then the triple (ζ,η,υ) gives corresponding coordinates for the space $N(W)$ and on $N(W)$ we have the relation $\upsilon=\lambda\zeta+\mu\eta$, for unique scalar functions λ and μ, defined globally on the space $N(W)$. Put $\theta\equiv\lambda\,\mathrm{d}\mu-\mu\,\mathrm{d}\lambda$. Denote by χ the holomorphic one form on $H(W)$, taking values in the space $\Omega^2_{\mathbb{C}}$, given by the formula: $\chi\equiv\upsilon\wedge\mathrm{d}\upsilon$. Then the form χ also represents a holomorphic one form globally defined on the space W, taking values in $\Omega^2_{\mathbb{C}}\otimes\Theta(2)$. Similarly, the functions λ and μ may be regarded as sections over $M(W)$ of the pullback of the sheaf $\Theta(1)$ along the map w. Also the form θ may be regarded as a holomorphic one form on the space $M(W)$, taking values in the pullback of the sheaf $\Theta(2)$ along the map w. Then we have the relation, valid for any $(\zeta,\eta)\in L(W)$:

$$\chi(\zeta,\eta)=\zeta\wedge\eta\Theta(\zeta,\eta). \tag{7.2}$$

One may represent the $(0,1)$ form β as $\beta=\iota((B(\upsilon,\overline{\upsilon}))\,\mathrm{d}\overline{\upsilon}$, where B is a smooth $V^*_{\mathbb{C}}$-valued function, globally defined on the space $H(W)$ and homogeneous of degrees $(-2,-1)$ in the pair $(\upsilon,\overline{\upsilon})$, such that $\iota(B(\upsilon,\overline{\upsilon}))\overline{\upsilon}=0$. Henceforth we abbreviate $B(\upsilon,\overline{\upsilon})$ by $B(\upsilon)$.

Now the product $\beta\chi$, being homogeneous of degrees $(0,0)$ in the pair $(\upsilon,\overline{\upsilon})$ represents a smooth $(1,1)$ form defined on the space W, taking values in $\Omega^2_{\mathbb{C}}$. Put $\Gamma(\beta)\equiv w^*(\beta\chi)$ and $\gamma(\beta)\equiv w^*(b)\theta$, where w^* is the pullback along the map w. Then $\Gamma(\beta)$ and $\gamma(\beta)$ are $(1,1)$ forms on $M(W)$, taking values in $\Omega^2_{\mathbb{C}}$ and $\mathbb{C}$, respectively. Therefore, the quantities $\Phi(\beta)\equiv L(\Gamma(\beta))$ and $\phi(\beta)\equiv L(\gamma(\beta))$ are well-defined smooth functions on $L(W)$, taking values in the spaces $\Omega^2_{\mathbb{C}}$ and $\mathbb{C}$, respectively. Further, from equation (7.2), we have the relation $\Phi(\beta)=(\zeta\wedge\eta)\phi(\beta)$.

Lemma 7.1. *If the form β is $\overline{\partial}$-exact, then the functions $\Phi(\beta)$ and $\phi(\beta)$ vanish identically.*

Proof. Fix any $(\zeta,\eta)\in L(W)$. We only need to verify that $\Phi(\beta)(\zeta,\eta)$ vanishes, when $\beta=\overline{\partial}\alpha$, for some α. If $\beta=\overline{\partial}\alpha$, then we have $\beta\chi=\overline{\partial}(\alpha\chi)$, since the form χ is holomorphic. Since the restriction map to the space $M(\zeta,\eta)$ is holomorphic, we have, for any $(\zeta,\eta)\in L(W)$ the following relations:

$$\begin{aligned}(\beta\chi)(\zeta,\eta)&=[\overline{\partial}(\alpha\chi)](\zeta,\eta)=\overline{\partial}[(\alpha\chi)(\zeta,\eta)]\\&=(\mathrm{d}-\partial)[(\alpha\chi)(\zeta,\eta)]=\mathrm{d}[(\alpha\chi)(\zeta,\eta)].\end{aligned} \tag{7.3}$$

In equation (7.3), we have used the fact that, since the form $(\alpha\chi)(\zeta,\eta)$ is of type $(1,0)$, defined on the one-dimensional complex manifold, $M(\zeta,\eta)$, it is killed by the

∂-operator. Equation (7.3) then shows that the form $(\beta\chi)(\zeta,\eta)$ is exact and therefore represents the zero element of the cohomology group $H^2(M(\zeta,\eta),\Omega^2_{\mathbb{C}})$. Hence from the definition of $\Phi(\beta)$, we have $\Phi(\beta)(\zeta,\eta)=0$, so the functions $\Phi(\beta)$ and $\phi(\beta)$ vanish identically, as required. ■

We next derive explicit expressions for the quantities $\phi(\beta)$ and $\Phi(\beta)$. It is sufficient to give a formula for just the quantity $\phi(\beta)$. We may use the ratios $\rho=\mu/\lambda$ and $\sigma=\lambda/\mu$ as (extended) complex coordinates for the Riemann sphere. These ratios are related by the formula $\rho\sigma=1$.

One has the following formulas, at any fixed $(\zeta,\eta)\in L(W)$:

$$\upsilon=\lambda(\zeta+\rho\eta)=\mu(\sigma\zeta+\eta); \tag{7.4}$$

$$\theta=\lambda^2\,\mathrm{d}\rho=-\mu^2\,\mathrm{d}\sigma; \tag{7.5}$$

$$\begin{aligned}\beta&=\iota(B(\upsilon))(\mathrm{d}\overline{\upsilon})=\iota(B(\upsilon))(\mathrm{d}[\overline{\lambda}(\overline{\zeta}+\overline{\rho\eta})])=\overline{\lambda}\iota(B(\upsilon))d[(\overline{\zeta}+\overline{\rho\eta})]\\&=\overline{\lambda}(\iota(B(\upsilon))(\overline{\eta}))\,\mathrm{d}\overline{\rho}=\lambda^{-2}\iota(B(\zeta+\rho\eta))(\overline{\eta})\,\mathrm{d}\overline{\rho}=\iota(B(\upsilon))(\mathrm{d}[\overline{\mu}(\overline{\sigma}\overline{\zeta}+\overline{\eta})])\\&=\overline{\mu}\iota(B(\upsilon))(\mathrm{d}[(\overline{\sigma}\overline{\zeta}+\overline{\eta})])=\overline{\mu}(\iota(B(\upsilon))(\overline{\zeta}))\,\mathrm{d}\overline{\sigma}\\&=\mu^{-2}\iota(B(\sigma\zeta+\eta))(\overline{\zeta})\,\mathrm{d}\overline{\sigma}.\end{aligned} \tag{7.6}$$

In equation (7.6), we have used the fact that $\iota(B(\upsilon)\overline{\upsilon}=0$.

Then for any fixed $(\zeta,\eta)\in L(W)$, we have the following expressions for the form $\gamma(\beta)$, using the homogeneity of the function $B(\upsilon)$:

$$\begin{aligned}\gamma(\beta)=\beta\theta&=\overline{\lambda}\iota(B(\lambda\zeta+\mu\eta))(\overline{\eta})\,\mathrm{d}\overline{\rho}(\lambda^2\,\mathrm{d}\rho)=\iota(B(\zeta+\rho\eta))(\overline{\eta})\,\mathrm{d}\overline{\rho}\,\mathrm{d}\rho\\&=\overline{\mu}\iota(B(\lambda\zeta+\mu\eta))(\overline{\zeta})\,\mathrm{d}\overline{\sigma}(-\mu^2\,\mathrm{d}\sigma)=-\iota(B(\sigma\zeta+\eta))(\overline{\zeta})\,\mathrm{d}\overline{\sigma}\,\mathrm{d}\sigma.\end{aligned} \tag{7.7}$$

Note that the function $B(\zeta+\rho\eta)$ is smooth on the domain of all $(\zeta,\eta,\rho)\in L(W)\times C$ and the function $B(\sigma\zeta+\eta)$ is smooth on the domain of all $(\zeta,\eta,\sigma)\in L(W)\times C$ and these functions are related, by the following formulas, valid provided $\rho\neq 0$ and $\sigma\neq 0$:

$$B(\sigma\zeta+\eta)=\rho^2\overline{\rho}B(\zeta+\rho\eta),\qquad \iota(B(\sigma\zeta+\eta))(\overline{\zeta})=-(\rho\overline{\rho})^2\iota(B(\zeta+\rho\eta))(\overline{\eta}). \tag{7.8}$$

So now the integral giving the quantity $\phi(\beta)(\zeta,\eta)$ may be written:

$$\begin{aligned}\phi(\beta)(\zeta,\eta)&=\int_{M(\zeta,\eta)}[(\beta\theta)(\zeta,\eta)]\\&=\int\iota(B(\zeta+\rho\eta))(\overline{\eta})\,\mathrm{d}\overline{\rho}\,\mathrm{d}\rho=-\int\iota(B(\sigma\zeta+\eta))(\overline{\zeta})\,\mathrm{d}\overline{\zeta}\,\mathrm{d}\zeta.\end{aligned} \tag{7.9}$$

The integrals of equation (7.9) are taken over the whole complex plane, which is equipped with its usual orientation. We wish to avoid all questions of convergence of integrals at infinity, so we will rewrite equation (7.9) as a sum of manifestly finite integrals, using the fact that $|r|\geqslant 1$, if and only if $|s|\leqslant 1$. Denote by D the set of all complex numbers of modulus not more than unity, equipped with its standard orientation, and by ∂D its boundary, oriented counterclockwise.

Lemma 7.2. *We have the integral expression, valid for any* $(\zeta,\eta)\in L(W)$*:*

$$\phi(\beta)(\zeta,\eta)=\int_D[\iota(B(\zeta+\lambda\eta))(\overline{\eta})-\iota(B(\lambda\zeta+\eta))(\overline{\zeta})]\,\mathrm{d}\overline{\lambda}\,\mathrm{d}\lambda. \tag{7.10}$$

Proof. We take either of the integral expressions of equation (7.9) and perform an inversion on the part of the integral outside the unit disc D. The required result then follows immediately, after using equation (7.8). ∎

Equation (7.10) shows that the function $\phi(\beta)$ is globally defined and everywhere smooth on the space $L(W)$. Note that equations (7.8) and (7.10) are easily shown to imply that the quantity $\phi(\beta)(\zeta,\eta)$ is skew in its arguments:

$$\phi(\beta)(\zeta,\eta) = -\phi(\beta)(\eta,\zeta), \tag{7.11}$$

valid for all $(\zeta,\eta)\in L(W)$. In the following we shall use the chain rule frequently. We shall use the following abbreviations systematically:

1. $A\cdot B$ for the quantity $\iota(A)(B)$;
2. X for the variable $\zeta+\lambda\eta$ and Y for the variable $\lambda\zeta+\eta$ (where $\lambda\in\mathbb{C}$);
3. Z for the variable $\zeta+re^{i\theta}\eta$ and U for the variable $re^{i\theta}z+\eta$ (where r and θ are real);
4. ∂_λ, $\partial_{\overline{\lambda}}$, ∂_r and ∂_θ for the partial derivatives $\partial/\partial\lambda$, $\partial/\partial\overline{\lambda}$, $\partial/\partial r$ and $\partial/\partial\theta$, respectively.

Lemma 7.3. *The function $\phi(\beta)(\zeta,\eta)$ is holomorphic in the variables ζ and η.*

Proof. First, we differentiate equation (7.10) with respect to the variable $\overline{\eta}$. Using the chain rule and the fact that the form β is $\overline{\partial}$-closed, we have the following formulas, for any fixed $A\in V_{\mathbb{C}}$:

$$\begin{aligned}\partial_{\overline{\eta}}\cdot A(B(\zeta+\lambda\eta)\cdot\overline{\eta}) &= \overline{\lambda}(\partial_{\overline{X}}\cdot AB(X))\cdot\overline{\eta}+B(X)\cdot A\\ &= \overline{\lambda}(\partial_{\overline{X}}(B(X)\cdot A))\cdot\overline{\eta}+B(X)\cdot A\\ &= \overline{\lambda}\partial_{\overline{\lambda}}(B(X)\cdot A))+B(X)\cdot A\\ &= \partial_{\overline{\lambda}}[\overline{\lambda}B(X)\cdot A],\end{aligned} \tag{7.12}$$

$$\begin{aligned}\partial_{\overline{\eta}}\cdot A(B(\lambda\zeta+\eta)\cdot\overline{\zeta}) &= (\partial_{Y'}\cdot AB(Y))\cdot\overline{\zeta}\\ &= [\partial_{Y'}(B(Y)\cdot A)]\cdot\overline{\zeta} = \partial_{\overline{\lambda}}(B(Y)\cdot A),\end{aligned} \tag{7.13}$$

$$\begin{aligned}\partial_{\overline{\eta}}\phi(\beta)(\zeta,\eta) &= \int_D \partial_{\overline{\lambda}}[\overline{\lambda}(B(\zeta+\lambda\eta)-B(\lambda\zeta+\eta)]\,d\overline{\lambda}\,d\lambda\\ &= \int_D d[\overline{\lambda}B(\zeta+\lambda\eta)-B(\lambda\zeta+\eta)]\,d\lambda\\ &= \int_{\partial D}[\overline{\lambda}B(\zeta+\lambda\eta)-B(\lambda\zeta+\eta)]\,d\lambda\\ &= \int_{\partial D}[\lambda^{-1}B(\zeta+\lambda\eta)-\lambda^{-1}B(\zeta+\lambda^{-1}\eta)]\,d\lambda\\ &= \int_{\partial D}[\lambda^{-1}B(\zeta+\lambda\eta)]\,d\lambda-\int_{\partial D}[\mu B(\zeta+\mu\eta)]\mu^{-2}\,d\mu = 0.\end{aligned} \tag{7.14}$$

Hence the function $\phi(\beta)(\zeta,\eta)$ is holomorphic in the variable η. By the skew symmetry property of $\phi(\beta)$, equation (7.11), the function $\phi(\beta)$ must be holomorphic also in its dependence on the variable ζ, as required. ∎

Lemma 7.4. *The function $\phi(\beta)(\zeta,\eta)$ is homogeneous of degrees $(-1,-1)$ in the variables ζ and η.*

Proof. We differentiate equation (7.10), using the chain rule and equation (7.8):

$$\begin{aligned}(\partial_\zeta\phi(\beta)(\zeta,\eta))\cdot\zeta &= \int_D \partial_\zeta[B(\zeta+\lambda\eta)\cdot\overline{\eta} - B(\lambda\zeta+\eta)\cdot\overline{\zeta}]\cdot\zeta\,\mathrm{d}\overline{\lambda}\,\mathrm{d}\lambda \\ &= \int_D [(\partial_X B(X))\cdot\overline{\eta} - \lambda(\partial_Y B(Y))\cdot\overline{\zeta}]\cdot\zeta\,\mathrm{d}\overline{\lambda}\,\mathrm{d}\lambda \\ &= \int_D [\partial_X(B(X)\cdot\overline{\eta})\cdot(X-\lambda\eta) - \lambda\partial_\lambda(B(Y)\cdot\overline{\zeta})]\,\mathrm{d}\overline{\lambda}\,\mathrm{d}\lambda \\ &= \int_D [-2B(X)\cdot\overline{\eta} - \lambda\partial_\lambda(B(X)\cdot\overline{\eta} + B(Y)\cdot\overline{\zeta})]\,\mathrm{d}\overline{\lambda}\,\mathrm{d}\lambda \\ &= -\phi(\beta)(\zeta,\eta) - \int_D [\partial_\lambda(\lambda B(X)\cdot\overline{\eta} + \lambda B(Y)\cdot\overline{\zeta})]\,\mathrm{d}\overline{\lambda}\,\mathrm{d}\lambda \\ &= -\phi(\beta)(\zeta,\eta) + \int_{\partial D} [\lambda B(\zeta+\lambda\eta)\cdot\overline{\eta} + \lambda B(\lambda\zeta+\eta)\cdot\overline{\zeta}]\,\mathrm{d}\overline{\lambda} \\ &= -\phi(\beta)(\zeta,\eta) + \int_{\partial D} [-\overline{\lambda}B(\zeta+\lambda\eta)\cdot\overline{\eta}\,\mathrm{d}\lambda - \lambda B(\zeta+\overline{\lambda}\eta)\cdot\overline{\eta}\,\mathrm{d}\overline{\lambda}] \\ &= -\phi(\beta)(\zeta,\eta). \end{aligned} \tag{7.15}$$

So the function $\phi(\beta)$ is homogeneous of degree minus one in the variable ζ. By skew symmetry, equation (7.11), the function $\phi(\beta)$ is also homogeneous of degree minus one in the variable η, as required. ∎

Next consider the form $\xi(\beta) \equiv \upsilon\beta\theta$. Then for any $(\zeta,\eta) \in L(W)$, the form $\xi(\beta)(\zeta,\eta)$ is a smooth $(1,1)$ form, defined globally on the space $M(\zeta,\eta)$ and homogeneous of degrees $(-1,0,-1,0)$ in the variables $(\zeta,\overline{\zeta},\eta,\overline{\eta})$, taking values in the sheaf $\Theta(1)\otimes V_{\mathbb{C}}$. So the form $\xi(\beta)(\zeta,\eta)$ represents an element of the sheaf cohomology group $H^{1,1}[M(\zeta,\eta),\Theta(1)]\otimes V_{\mathbb{C}}$.

Lemma 7.5. *If ω is a $(1,1)$ form on the complex projective space $P_{\mathbb{C}}(X)$ of X, a two-dimensional complex vector space, taking values in the sheaf $\Theta(1)$, then $\omega = \overline{\partial}\psi$, for a unique $(1,0)$ form ψ on $P_{\mathbb{C}}(X)$.*

Proof. The form ω represents an element of the sheaf cohomology group

$$H^{1,1}(P_{\mathbb{C}}(X),\Theta(1)).$$

By Serre duality the group $H^{1,1}(P_{\mathbb{C}}(X),\Theta(1))$ is dual to the group

$$H^{0,0}(P_{\mathbb{C}}(X),\Theta(-1)),$$

which is the space of global holomorphic sections of the sheaf $\Theta(-1)$. But by Liouville's theorem, the only such section is the zero section. So both the cohomology groups $H^{0,0}(P_{\mathbb{C}}(X),\Theta(-1))$ and $H^{1,1}(P_{\mathbb{C}}(X),\Theta(1))$ vanish. Therefore, the form ω is $\overline{\partial}$-exact: a smooth $(1,0)$ form, ψ, taking values in $\Theta(1)$, exists such that $\omega = \overline{\partial}\psi$. The ambiguity in the quantity ψ is measured by the cohomology group $H^{1,0}(P_{\mathbb{C}}(X),\Theta(1))$. But this group is in turn isomorphic to the group $H^{0,0}(P_{\mathbb{C}}(X),\Theta(-1))\otimes\Omega^2(X^*)$, since the space $P_{\mathbb{C}}(X)$ possesses a global everywhere non-vanishing holomorphic section of the bundle of $(1,0)$ forms with values in the bundle $\Theta(2)\otimes\Omega^2(X^*)$. Since

$H^{0,0}(P_{\mathbb{C}}(X), \Theta(-1))$ vanishes, we have $H^{1,0}(P_{\mathbb{C}}(X), \Theta(1)) = 0$ and the form ψ is unique. ∎

Applying lemma 7.5, we see that the form $\xi(\beta)(\zeta, \eta)$ is $\overline{\partial}$-exact: $\xi(\beta)(\zeta, \eta) = \overline{\partial}(\rho(\beta)(\zeta, \eta))$, for a unique $(1, 0)$ form $\rho(\beta)(\zeta, \eta)$. We find an explicit formula for $\rho(\beta)$, essentially by the standard method of Grothendieck, as follows. Define the form $\omega(\beta)$ by the formula: $\omega(\beta) \equiv \mu^{-1}\xi(\beta)$. Then the form $\omega(\beta)$ is a $(1, 1)$ form, smooth everywhere on $M(W)$, except where $\mu = 0$ and is homogeneous of degrees $(-1, 0)$ in the variables $(\zeta, \overline{\zeta})$ and $(0, 0)$ in the variables $(\eta, \overline{\eta})$.

The forms $\xi(\beta)(\zeta, \eta)$ and $\omega(\beta)(\zeta, \eta)$ are given explicitly by the formulas,

$$\xi(\beta)(\zeta, \eta) = \mu\omega(\beta)(\zeta, \eta), \tag{7.16}$$

$$\begin{aligned} \omega(\beta)(\zeta, \eta) &= \rho^{-1}(\zeta + \rho\eta)B(\zeta + \rho\eta) \cdot \overline{\eta}\, \mathrm{d}\overline{\rho}\, \mathrm{d}\rho \\ &= -(\sigma\zeta + \eta)B(\sigma\zeta + \eta) \cdot \overline{\zeta}\, \mathrm{d}\overline{\sigma}\, \mathrm{d}\sigma. \end{aligned} \tag{7.17}$$

Lemma 7.6. *The integral of the form $\omega(\beta)(\zeta, \eta)$ over the oriented two sphere $M(\zeta, \eta)$, which represents the positive generator of the integral second homology group of $M(\zeta, \eta)$, is well defined and smooth in the variables (ζ, η).*

Proof. We construct an explicit formula for this integral. Denote the value of this integral, if it exists, by $\alpha(\zeta, \eta)$. Using equation (7.17), the required integral may be written as follows:

$$\begin{aligned} \alpha(\zeta, \eta) &= \int_{M(\zeta,\eta)} \omega(\beta)(\zeta, \eta) = \int \rho^{-1}(\zeta + \rho\eta)(B(\zeta + \rho\eta) \cdot \overline{\eta})\, \mathrm{d}\overline{\rho}\, \mathrm{d}\rho \\ &= -\int (\sigma\zeta + \eta)(B(\sigma\zeta + \eta) \cdot \overline{\zeta})\, \mathrm{d}\overline{\sigma}\, \mathrm{d}\sigma = \eta\phi(\beta)(\zeta, \eta) + \zeta\psi(\zeta, \eta), \end{aligned} \tag{7.18}$$

$$\psi(\zeta, \eta) \equiv \int [B(\zeta + \rho\eta) \cdot \overline{\eta}]\rho^{-1}\, \mathrm{d}\overline{\rho}\, \mathrm{d}\rho = -\int [B(\sigma\zeta + \eta) \cdot \overline{\zeta}]\sigma\, \mathrm{d}\overline{\sigma}\, \mathrm{d}\sigma. \tag{7.19}$$

Here each integral is to be taken over the whole complex plane, with its usual orientation.

Splitting the integral for $\psi(\zeta, \eta)$ into two pieces, we have also

$$\psi(\zeta, \eta) = \psi_1(\zeta, \eta) - \psi_2(\zeta, \eta), \tag{7.20}$$

$$\psi_1(\zeta, \eta) = \int_D [B(\zeta + \lambda\eta) \cdot \overline{\eta}]\lambda^{-1}\, \mathrm{d}\overline{\lambda}\, \mathrm{d}\lambda, \tag{7.21}$$

$$\psi_2(\zeta, \eta) = \int_D [B(\lambda\zeta + \eta) \cdot \overline{\zeta}]l\, \mathrm{d}\overline{\lambda}\, \mathrm{d}\lambda. \tag{7.22}$$

Now the integral giving the quantity $\psi_2(\zeta, \eta)$ is clearly convergent and smooth in the quantities ζ and η. Equation (7.21) may be rewritten as the following multiple integral, using polar coordinates:

$$\psi_1(\zeta, \eta) = 2\mathrm{i} \int_{-\pi}^{\pi} \int_0^1 [B(\zeta + r\mathrm{e}^{\mathrm{i}\theta}\eta) \cdot \overline{\eta}\mathrm{e}^{-\mathrm{i}\theta}]\, \mathrm{d}r\, \mathrm{d}\theta. \tag{7.23}$$

In this form it is obviously convergent and smooth in the parameters ζ and η. The lemma is proved. ∎

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Next we study the various derivatives of the quantity $\alpha(\zeta,\eta)$. First consider the derivative $(\partial_{\overline{\zeta}}\psi_2)\cdot\overline{\eta}$. Using the chain rule, we have, since the function $B(Y)$ is homogeneous of degrees $(-2,-1)$ in the pair (Y,Y'), for any constant $A\in V_{\mathbb{C}}$:

$$\begin{aligned}0&=\overline{\lambda}[\partial_{\overline{Y}}(B(Y)\cdot A)]\cdot\overline{Y}+\overline{\lambda}B(Y)\cdot A\\&=\overline{\lambda}[\partial_{\overline{Y}}(B(Y)\cdot A)]\cdot(\overline{\lambda\zeta}+\overline{\eta})+\overline{\lambda}B(Y)\cdot A\\&=[\partial_{\overline{\zeta}}(B(Y)\cdot A)]\cdot\overline{\eta}+\overline{\lambda}^2\partial_{\overline{\lambda}}(B(Y)\cdot A)+\overline{\lambda}B(Y)\cdot A.\end{aligned}\tag{7.24}$$

In particular, we have from equation (7.24), after replacing A by $\overline{\zeta}$, the relation:

$$\begin{aligned}0&=[\partial_{\overline{\zeta}}(B(Y)\cdot\overline{\zeta})]\cdot\overline{\eta}-B(Y)\cdot\overline{\eta}+\overline{\lambda}^2\partial_{\overline{\lambda}}(B(Y)\cdot\overline{\zeta})+\overline{\lambda}B(Y)\cdot\overline{\zeta}\\&=[\partial_{\overline{\zeta}}(B(Y)\cdot\overline{\zeta})]\cdot\overline{\eta}+\partial_{\overline{\lambda}}[\overline{\lambda}^2(B(Y)\cdot\overline{\zeta})]-B(Y)\cdot(\overline{\eta}+\overline{\lambda\zeta})\\&=[\partial_{\overline{\zeta}}(B(Y)\cdot\overline{\zeta}]\cdot\overline{\eta}+\partial_{\overline{\lambda}}[\overline{\lambda}^2(B(Y)\cdot\overline{\zeta})].\end{aligned}\tag{7.25}$$

Using equations (7.22) and (7.25), we have the following derivative:

$$\begin{aligned}[\partial_{\overline{\zeta}}\psi_2(\zeta,\eta)]\cdot\overline{\eta}&=\int_D([\partial_{\overline{\zeta}}(B(Y)\cdot\overline{\zeta})]\cdot\overline{\eta})\lambda\,\mathrm{d}\overline{\lambda}\,\mathrm{d}\lambda\\&=-\int_D\mathrm{d}[\lambda\overline{\lambda}^2(B(Y)\cdot\overline{\zeta})\,\mathrm{d}\lambda]=A_1,\end{aligned}\tag{7.26}$$

$$A_1\equiv-\int_{\partial D}[\overline{\lambda}B(\lambda\zeta+\eta)\cdot\overline{\zeta}\,\mathrm{d}\lambda.\tag{7.27}$$

Next consider the derivative $(\partial_{\overline{\zeta}}\psi_1)\cdot\overline{\eta}$. If $r>0$, we have, using the chain rule,

$$\tfrac{1}{2}\mathrm{e}^{\mathrm{i}\theta}(\partial/\partial r+\mathrm{i}r^{-1}\partial/\partial\theta)B(\zeta+r\mathrm{e}^{\mathrm{i}\theta}\eta)=[(\partial_{\overline{Z}}B)(Z)]\cdot\overline{\eta}=(\partial_{\overline{\zeta}}B)(Z)\cdot\overline{\eta}.\tag{7.28}$$

Define an auxiliary function f by the formula:

$$rf(\zeta,\eta,r,\theta)=B(\zeta+r\mathrm{e}^{\mathrm{i}\theta}\eta)-B(\zeta).\tag{7.29}$$

The function f is defined by equation (7.29), for $r\neq0$ and by Taylor's theorem extends smoothly to the surface $r=0$. Then we may rewrite equation (7.28), as follows, valid everywhere:

$$\tfrac{1}{2}\mathrm{e}^{\mathrm{i}\theta}(\partial_r r+\mathrm{i}\partial_\theta)f(\zeta,\eta,r,\theta)=(\partial_{\overline{\zeta}}B)(\zeta+r\mathrm{e}^{\mathrm{i}\theta}\eta)\cdot\overline{\eta}.\tag{7.30}$$

Using equations (7.8), (7.23) and (7.30), we find the following derivative:

$$\begin{aligned}[\partial_{\overline{\zeta}}\psi_1(\zeta,\eta)]\cdot\overline{\eta}&=2\mathrm{i}\int_{-\pi}^{\pi}\int_0^1(\partial_{\overline{\zeta}}[B(\zeta+r\mathrm{e}^{\mathrm{i}\theta}\eta)\cdot\overline{\eta}])\cdot\overline{\eta}\mathrm{e}^{-\mathrm{i}\theta}\,\mathrm{d}r\,\mathrm{d}\theta\\&=\mathrm{i}\int_{-\pi}^{\pi}\int_0^1(\partial_r r+\mathrm{i}\partial_\theta)[f(\zeta,\eta,r,\theta)\cdot\overline{\eta}]\,\mathrm{d}r\,\mathrm{d}\theta\\&=\mathrm{i}\int_{-\pi}^{\pi}[f(\zeta,\eta,1,\theta)\cdot\overline{\eta}]\,\mathrm{d}\theta=\mathrm{i}\int_{-\pi}^{\pi}[B(\zeta+\mathrm{e}^{\mathrm{i}\theta}\eta)\cdot\overline{\eta}-B(\zeta)\cdot\overline{\eta}]\,\mathrm{d}\theta\\&=-2\mathrm{i}\pi B(\zeta)\cdot\overline{\eta}+\mathrm{i}\int_{-\pi}^{\pi}[B(\zeta+\mathrm{e}^{\mathrm{i}\theta}\eta)\cdot\overline{\eta}]\,\mathrm{d}\theta\\&=-2\mathrm{i}\pi B(\zeta)\cdot\overline{\eta}-\mathrm{i}\int_{-\pi}^{\pi}[B(\mathrm{e}^{-\mathrm{i}\theta}\zeta+\eta)\cdot\overline{\zeta}]\,\mathrm{d}\theta\\&=-2\mathrm{i}\pi B(\zeta)\cdot\overline{\eta}-\int_{\partial D}[B(\lambda\zeta+\eta)\cdot\overline{\zeta}]\overline{\lambda}\,\mathrm{d}\lambda=-2\mathrm{i}\pi B(\zeta)\cdot\overline{\eta}+A_1.\end{aligned}\tag{7.31}$$

So now, putting together equations (7.20), (7.26), (7.27) and (7.31), we have

$$[\partial_{\overline{\zeta}}\psi(\zeta,\eta)]\cdot\overline{\eta} = [\partial_{\overline{\zeta}}\psi_1(\zeta,\eta)]\cdot\overline{\eta} - [\partial_{\overline{\zeta}}\psi_2(\zeta,\eta)]\cdot\overline{\eta} = -2\mathrm{i}\pi B(\zeta)\cdot\overline{\eta}. \tag{7.32}$$

We may summarize as follows.

Lemma 7.7. *We have the following expression for the derivative* $(\partial_{\overline{\zeta}}\alpha)\cdot\overline{\eta}$*:*

$$[\partial_{\overline{\zeta}}\alpha(\zeta,\eta)]\cdot\overline{\eta} = -2\mathrm{i}\pi\zeta B(\zeta)\cdot\overline{\eta}. \tag{7.33}$$

Proof. We use equations (7.18) and (7.32), together with lemma 7.3. ■

Next we tackle the derivative of $\alpha(\zeta,\eta)$ with respect to the variable $\overline{\eta}$. By equation (7.18) and lemma 7.3, we have the formula: $\partial_{\overline{\eta}}\alpha = \zeta\partial_{\overline{\eta}}\psi$, so it is only necessary to calculate the quantity $\partial_{\overline{\eta}}\psi$. From equation (7.22), using the chain rule, we have

$$\begin{aligned}
\partial_{\overline{\eta}}\psi_2(\zeta,\eta) &= \int_D [\partial_{\overline{\eta}}B(\lambda\zeta+\eta)\cdot\overline{\zeta}]\lambda\,\mathrm{d}\overline{\lambda}\,\mathrm{d}\lambda \\
&= \int_D [\partial_{\overline{Y}}(B(Y)\cdot\overline{\zeta})]\lambda\,\mathrm{d}\overline{\lambda}\,\mathrm{d}\lambda = \int_D [(\partial_{\overline{Y}}\cdot\overline{\zeta})B(Y)]\lambda\,\mathrm{d}\overline{\lambda}\,\mathrm{d}\lambda \\
&= \int_D [\partial_{\overline{\lambda}}B(Y)]\lambda\,\mathrm{d}\overline{\lambda}\,\mathrm{d}\lambda \\
&= \int_{\partial D} B(\lambda\zeta+\eta)\lambda\,\mathrm{d}\lambda.
\end{aligned} \tag{7.34}$$

Note that in going from the second to the third line of equation (7.34), we have used that the form β is $\overline{\partial}$-closed. Next from equation (7.23), using the chain rule, we have

$$\begin{aligned}
\partial_{\overline{\eta}}\psi_1(\zeta,\eta) &= 2\mathrm{i}\int_{-\pi}^{\pi}\int_0^1 \partial_{\overline{\eta}}[B(\zeta+r\mathrm{e}^{\mathrm{i}\theta}\eta)\cdot\overline{\eta}\mathrm{e}^{-\mathrm{i}\theta}]\,\mathrm{d}r\,\mathrm{d}\theta \\
&= 2\mathrm{i}\int_{-\pi}^{\pi}\int_0^1 [r\mathrm{e}^{-2\mathrm{i}\theta}\partial_{\overline{Z}}B(Z)\cdot\overline{\eta} + B(Z)\mathrm{e}^{-\mathrm{i}\theta}]\,\mathrm{d}r\,\mathrm{d}\theta \\
&= 2\mathrm{i}\int_{-\pi}^{\pi}\int_0^1 [r\mathrm{e}^{-2\mathrm{i}\theta}(\partial_{\overline{Z}}\cdot\overline{\eta})(B(Z)) + B(Z)\mathrm{e}^{-\mathrm{i}\theta}]\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \mathrm{i}\int_{-\pi}^{\pi}\int_0^1 [(\partial_r r + \mathrm{i}\partial_\theta)B(Z)\mathrm{e}^{-\mathrm{i}\theta}]\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \mathrm{i}\int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}\theta}B(\zeta+\mathrm{e}^{\mathrm{i}\theta}\eta)\,\mathrm{d}\theta \\
&= \mathrm{i}\int_{-\pi}^{\pi} \mathrm{e}^{-2\mathrm{i}\theta}B(\mathrm{e}^{-\mathrm{i}\theta}\zeta+\eta)\,\mathrm{d}\theta \\
&= \int_{\partial D} B(\lambda\zeta+\eta)\lambda\,\mathrm{d}\lambda.
\end{aligned} \tag{7.35}$$

Again in going from the second to the third line of equation (7.35), we have used that the form β is $\overline{\partial}$-closed.

Theorem 7.8. *The functions* $\psi(\zeta,\eta)$ *and* $\alpha(\zeta,\eta)$ *are holomorphic in the variable* η.

Proof. From equations (7.20), (7.34) and (7.35), we find that the quantity $\psi(\zeta,\eta)$ is holomorphic in the variable η. Then, using equation (7.18) and lemma 7.3, we see that the quantity $\alpha(\zeta,\eta)$ is also holomorphic in the variable η. ■

Corollary 7.9. *The function $\psi(\zeta,\eta)$ obeys the differential equation,*

$$\partial_{\overline{\zeta}}\psi(\zeta,\eta) = -2\mathrm{i}\pi B(z). \tag{7.36}$$

Proof. Differentiate both sides of equation (7.32) with respect to the variable $\overline{\eta}$ and use theorem 7.8. ■

Lemma 7.10. *The quantity $\psi(\zeta,\eta)$ is homogeneous of degree zero in the variable η, i.e. $[\partial_\eta\psi(\zeta,\eta)]\cdot\eta = 0$.*

Proof. By equation (7.22), we have, using the chain rule,

$$\begin{aligned}
[\partial_\eta\psi_2(\zeta,\eta)]\cdot\eta &= \int_D [\partial_\eta(B(\lambda\zeta+\eta)\cdot\overline{\zeta})\cdot\eta]\lambda\,\mathrm{d}\overline{\lambda}\,\mathrm{d}\lambda \\
&= \int_D [\partial_Y(B(Y)\cdot\overline{\zeta})\cdot(Y-\lambda\zeta)]\lambda\,\mathrm{d}\overline{\lambda}\,\mathrm{d}\lambda \\
&= \int_D [(-2-\lambda\partial_\lambda)(B(Y)\cdot\overline{\zeta})]\lambda\,\mathrm{d}\overline{\lambda}\,\mathrm{d}\lambda \\
&= -\int_D [\partial_\lambda(\lambda^2 B(Y)\cdot\overline{\zeta})]\,\mathrm{d}\overline{\lambda}\,\mathrm{d}\lambda = \int_{\partial D}[\lambda^2 B(Y)\cdot\overline{\zeta}]\,\mathrm{d}\overline{\lambda} \\
&= -\int_{\partial D}[B(\lambda\zeta+\eta)\cdot\overline{\zeta}]\,\mathrm{d}\lambda.
\end{aligned} \tag{7.37}$$

By equation (7.23), we have

$$\begin{aligned}
[\partial_\eta\psi_1(\zeta,\eta)]\cdot\eta &= 2\mathrm{i}\int_{-\pi}^{\pi}\int_0^1 [\partial_\eta(B(\zeta+r\mathrm{e}^{\mathrm{i}\theta}\eta)\cdot\overline{\eta})\mathrm{e}^{-\mathrm{i}\theta}]\cdot\eta\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \mathrm{i}\int_{-\pi}^{\pi}\int_0^1 [(\partial_r r - \mathrm{i}\partial_\theta)(\mathrm{e}^{-\mathrm{i}\theta}(B(\zeta+r\mathrm{e}^{\mathrm{i}\theta}\eta))(\overline{\eta}))]\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \mathrm{i}\int_{-\pi}^{\pi}[\mathrm{e}^{-\mathrm{i}\theta}B(\zeta+\mathrm{e}^{\mathrm{i}\theta}\eta)\cdot\overline{\eta}]\,\mathrm{d}\theta = -\mathrm{i}\int_{-\pi}^{\pi}[\mathrm{e}^{-\mathrm{i}\theta}B(\mathrm{e}^{-\mathrm{i}\theta}\zeta+\eta)\cdot\overline{\zeta}]\,\mathrm{d}\theta \\
&= -\int_{\partial D}[B(\lambda\zeta+\eta)\cdot\overline{\zeta}]\,\mathrm{d}\lambda.
\end{aligned} \tag{7.38}$$

Putting equations (7.20), (7.37) and (7.38) together now gives the required result. ■

Lemma 7.11. *The quantity $\psi(\zeta,\eta)$ is homogeneous of degree zero in the variable $\overline{\zeta}$, i.e. $[\partial_{\overline{\zeta}}\psi(\zeta,\eta)]\cdot\overline{\zeta} = 0$.*

Proof. By equation (7.36), we have, $[\partial_{\overline{\zeta}}\psi(\zeta,\eta)]\cdot\overline{\zeta} = -2\mathrm{i}\pi B(\zeta)\cdot\overline{\zeta} = 0$, as required. ■

Lemma 7.12. *The quantity $\psi(\zeta,\eta)$ is homogeneous of degree minus two in the variable ζ, i.e. $[\partial_\zeta\psi(\zeta,\eta)]\cdot\zeta = -2\psi(\zeta,\eta)$.*

Proof. First, we have the following derivative, using equations (7.8) and (7.22) and the chain rule:

$$\begin{aligned}[\partial_\zeta \psi_2(\zeta,\eta)]\cdot\zeta &= \int_D [\partial_\zeta(B(\lambda\zeta+\eta)\cdot\overline{\zeta})\cdot\zeta]\lambda\,\mathrm{d}\overline{\lambda}\,\mathrm{d}\lambda \\ &= \int_D [\partial_Y(B(Y)\cdot\overline{\zeta})\cdot\zeta]\lambda^2\,\mathrm{d}\overline{\lambda}\,\mathrm{d}\lambda \\ &= \int_D [\partial_\lambda(B(Y)\cdot\overline{\zeta})]\lambda^2\,\mathrm{d}\overline{\lambda}\,\mathrm{d}\lambda \\ &= -2\psi_2(\zeta,\eta) - \int_D d[\lambda^2 B(Y)\cdot\overline{\zeta}\,\mathrm{d}\overline{\lambda}] \\ &= -2\psi_2(\zeta,\eta) - \int_{\partial D}[\lambda^2 B(\lambda\zeta+\eta)\cdot\overline{\zeta}]\,\mathrm{d}\overline{\lambda} \\ &= -2\psi_2(\zeta,\eta) + \int_{\partial D} B(\lambda\zeta+\eta)\cdot\overline{\zeta}\,\mathrm{d}\lambda. \end{aligned} \tag{7.39}$$

Next we have the following derivative, using equation (7.23) and the chain rule:

$$\begin{aligned}[\partial_\zeta \psi_1(\zeta,\eta)]\cdot\zeta &= 2\mathrm{i}\int_{-\pi}^{\pi}\int_0^1 [\partial_\zeta(B(\zeta+r\mathrm{e}^{\mathrm{i}\theta}\eta)\cdot\overline{\eta})\mathrm{e}^{-\mathrm{i}\theta}]\cdot\zeta\,\mathrm{d}r\,\mathrm{d}\theta \\ &= 2\mathrm{i}\int_{-\pi}^{\pi}\int_0^1 [\partial_Z(B(Z)\cdot\overline{\eta})\mathrm{e}^{-\mathrm{i}\theta}]\cdot\zeta\,\mathrm{d}r\,\mathrm{d}\theta \\ &= 2\mathrm{i}\int_{-\pi}^{\pi}\int_0^1 [\partial_Z(B(Z)\cdot\overline{\eta})\mathrm{e}^{-\mathrm{i}\theta}]\cdot(Z-r\mathrm{e}^{\mathrm{i}\theta}\eta)\,\mathrm{d}r\,\mathrm{d}\theta \\ &= -2\psi_1(\zeta,\eta) - \mathrm{i}\int_{-\pi}^{\pi}\int_0^1\left[\left(\frac{\partial}{\partial r}r - \mathrm{i}\frac{\partial}{\partial\theta}\right)(\mathrm{e}^{-\mathrm{i}\theta}B(Z)\cdot\overline{\eta})\right]\mathrm{d}r\,\mathrm{d}\theta \\ &= -2\psi_1(\zeta,\eta) - \mathrm{i}\int_{-\pi}^{\pi}[\mathrm{e}^{-\mathrm{i}\theta}B(\zeta+\mathrm{e}^{\mathrm{i}\theta}\eta)\cdot\overline{\eta}]\,\mathrm{d}\theta \\ &= -2\psi_1(\zeta,\eta) + \mathrm{i}\int_{-\pi}^{\pi}[\mathrm{e}^{-\mathrm{i}\theta}B(\mathrm{e}^{-\mathrm{i}\theta}\zeta+\eta)\cdot\overline{\zeta}]\,\mathrm{d}\theta \\ &= -2\psi_1(\zeta,\eta) + \int_{\partial D}[B(\lambda\zeta+\eta)\cdot\overline{\zeta}]\,\mathrm{d}\lambda. \end{aligned} \tag{7.40}$$

Putting equations (7.20), (7.39) and (7.40) together now gives the required result. ■

Corollary 7.13. *The quantity* $\alpha(\zeta,\eta)$ *is homogeneous of degrees* $(-1,0,0,0)$ *in the variables* $(\zeta,\eta,\overline{\zeta},\overline{\eta})$.

Proof. The required result follows immediately from equation (7.18), together with lemmas 7.3, 7.4, 7.10, 7.11 and 7.12. ■

Equation (7.33) gives in particular the following formulas, using equation (7.17) and corollary 7.13:

$$\partial_{\overline{\rho}}\alpha(\zeta+\rho\eta,\eta) = -2\mathrm{i}\pi(\zeta+\rho\eta)B(\zeta+\rho\eta)\cdot\overline{\eta}, \tag{7.41}$$

$$\begin{aligned}(\overline{\partial}\alpha)(\lambda\zeta+\mu\eta,\eta)\theta &= \lambda(\overline{\partial}\alpha)(\zeta+\rho\eta,\eta)\,\mathrm{d}\rho \\ &= -2\mathrm{i}\pi\lambda(\zeta+\rho\eta)B(\zeta+\rho\eta)\cdot\overline{\eta}\,\mathrm{d}\overline{\rho}\,\mathrm{d}\rho = -2\mathrm{i}\pi\xi(\beta). \end{aligned} \tag{7.42}$$

Corollary 7.14. *The form* $\rho_{(\zeta,\eta)}(\beta) \equiv (-2\mathrm{i}\pi)^{-1}\alpha(\lambda\zeta + \mu\eta, \eta)\theta$ *satisfies for each fixed* $(\zeta, \eta) \in L(W)$ *the equation:*

$$\overline{\partial}\rho_{(\zeta,\eta)}(\beta) = \xi(\beta)(\zeta, \eta). \tag{7.43}$$

Proof. Equation (7.43) follows immediately from equation (7.42). ■

Next we are able to prove the following important corollary of theorem 7.8.

Corollary 7.15. *The sheaf cohomology group* $H^1(P_{\mathbb{C}}, \Theta(-2))$ *vanishes.*

Proof. Take the domain W to be all of $P_{\mathbb{C}}$. It is clear that W is admissible. For each fixed $\zeta \in V'_{\mathbb{C}}, \psi(\zeta, \eta)$ is defined and holomorphic in η, for all $\eta \in V_{\mathbb{C}}$, such that $\zeta \wedge \eta \neq 0$. But by lemma 7.10, the function $\psi(\zeta, \eta)$ is homogeneous of degree zero in η, so the function $\psi(\zeta, \eta)$ induces, for each fixed $\zeta \in V_{\mathbb{C}}$, a holomorphic function on the projective space $P_{\mathbb{C}}$, defined everywhere on $P_{\mathbb{C}}$, except at the point $p_{\mathbb{C}}(\zeta)$. But every such holomorphic function is constant. Therefore, the quantity $\psi(\zeta, \eta)$ does not depend on the variable η, so there exists a smooth function $\alpha(\zeta)$, defined globally on $V_{\mathbb{C}}$, such that $\alpha(\zeta) = \psi(\zeta, \eta)$, whenever $(\zeta, \eta) \in L(P_{\mathbb{C}})$. Equation (7.36) now reads

$$\partial_{\overline{\zeta}}\alpha(\zeta) = -2\mathrm{i}\pi B(\zeta). \tag{7.44}$$

Furthermore, since the function $\psi(\zeta, \eta)$ is homogeneous of degrees $(-2, 0)$ in the variables $(\zeta, \overline{\zeta})$, so is the function $\alpha(\zeta)$.

Therefore, the function $\alpha(\zeta)$ represents a smooth section of the sheaf $\Theta(-2)$, globally defined on the space $P_{\mathbb{C}}$. Then equation (7.44) shows that the $(0, 1)$ form, β, on the space $P_{\mathbb{C}}$, represented by the function $B(\zeta)$ is $\overline{\partial}$-exact. Since this is true for any such form β, the corollary is proved. ■

Now we return to a general admissible domain W. Here we cannot expect that $\psi(\zeta, \eta)$ is independent of η, since otherwise the cohomology group $H^1(W, \Theta(-2))$ would vanish, which is not true in general. However, we may instead just calculate directly the derivative of $\psi(\zeta, \eta)$ with respect to η. From equation (7.22), we have

$$\begin{aligned}\partial_\eta\psi_2(\zeta, \eta) &= \int_D [\partial_\eta B(\lambda\zeta + \eta) \cdot \overline{\zeta}]\lambda\, \mathrm{d}\overline{\lambda}\, \mathrm{d}\lambda \\ &= \int_D [\partial_Y (B(Y) \cdot \overline{\zeta})]\lambda\, \mathrm{d}\overline{\lambda}\, \mathrm{d}\lambda = \partial_\zeta \int_D [B(\lambda\zeta + \eta) \cdot \overline{\zeta}]\, \mathrm{d}\overline{\lambda}\, \mathrm{d}\lambda.\end{aligned} \tag{7.45}$$

Next we have, from equation (7.23),

$$\begin{aligned}\partial_\eta\psi_1(\zeta, \eta) &= 2\mathrm{i}\int_{-\pi}^{\pi}\int_0^1 [\partial_\eta(B(\zeta + r\mathrm{e}^{\mathrm{i}\theta}\eta))(\overline{\eta})\mathrm{e}^{-\mathrm{i}\theta}]\, \mathrm{d}r\, \mathrm{d}\theta \\ &= 2\mathrm{i}\int_{-\pi}^{\pi}\int_0^1 [\partial_Z(B(Z) \cdot \overline{\eta})r\, \mathrm{d}r\, \mathrm{d}\theta \\ &= 2\mathrm{i}\partial_\zeta\int_{-\pi}^{\pi}\int_0^1 [(B(\zeta + r\mathrm{e}^{\mathrm{i}\theta}\eta) \cdot \overline{\eta})r\, \mathrm{d}r\, \mathrm{d}\theta \\ &= \partial_\zeta \int_D [B(\zeta + \lambda\eta) \cdot \overline{\eta}]\, \mathrm{d}\overline{\lambda}\, \mathrm{d}\lambda.\end{aligned} \tag{7.46}$$

So we have the following lemma.

Lemma 7.16. *The quantities $\psi(\zeta,\eta)$ and $\phi(\beta)(\zeta,\eta)$ are related by the formula, valid for all $(\zeta,\eta)\in L(W)$:*

$$\partial_\eta\psi(\zeta,\eta)=\partial_\zeta\phi(\beta)(\zeta,\eta). \tag{7.47}$$

Proof. Combine equations (7.10), (7.20), (7.45) and (7.46). This gives the result immediately. ∎

Corollary 7.17. *If $\phi(\beta)$ vanishes identically, then the form B is $\overline{\partial}$-exact.*

Proof. When $\phi(\beta)=0$, by equation (7.47) and by theorem 7.8, the quantity $\psi(\zeta,\eta)$ has vanishing derivatives with respect to the variables η and $\overline{\eta}$. Therefore, it is locally constant in the variable η, for each fixed ζ. By definition of admissibility of the domain W, it then follows that the quantity $\psi(\zeta,\eta)$ is independent of η, so there exists a function $\alpha(\zeta)$, globally defined on W, such that $\alpha(\zeta)=\psi(\zeta,\eta)$, whenever $(\zeta,\eta)\in L(W)$. The rest of the proof follows exactly the argument of corollary 7.15. ∎

We have now established the following theorem, which combines many of the various results of this section.

Theorem 7.18. *Let β be a smooth $\overline{\partial}$-closed $(0,1)$ form with values in $\Theta(-2)$, defined on an admissible open set W in $P_{\mathbb{C}}$. Then the form β is represented by a smooth function, $B(\upsilon)$, taking values in the space $V^*_{\mathbb{C}}$, globally defined on the space $H(W)$, such that the function $B(\upsilon)$ is homogeneous of degrees $(-2,-1)$ in the variables $(\upsilon,\overline{\upsilon})$ and obeys the equations $B(\upsilon)\cdot\overline{\upsilon}=0$ and $\partial_{\overline{\upsilon}}\wedge B(\upsilon)=0$. Define the functions $\phi(\beta)$ and $\psi(\beta)$ on the space $L(W)$, by the formulas, valid for any $(\zeta,\eta)\in L(W)$:*

$$\phi(\beta)(\zeta,\eta)\equiv(2\mathrm{i}\pi)^{-1}\int_D[B(\zeta+\lambda\eta)\cdot\overline{\eta}-B(\lambda\zeta+\eta)\cdot\overline{\zeta}]\,\mathrm{d}\lambda\,\mathrm{d}\overline{\lambda}, \tag{7.48}$$

$$\psi(\beta)(\zeta,\eta)\equiv-\pi^{-1}\int_{-\pi}^{\pi}\int_0^1[B(\zeta+r\mathrm{e}^{\mathrm{i}\theta}\eta)+rB(r\zeta+\mathrm{e}^{\mathrm{i}\theta}\eta)]\cdot\overline{\eta}\mathrm{e}^{-\mathrm{i}\theta}\,\mathrm{d}r\,\mathrm{d}\theta. \tag{7.49}$$

Then the quantities $\phi(\beta)(\zeta,\eta)$ and $\psi(\beta)(\zeta,\eta)$ are each homogeneous in the variables $(\zeta,\eta,\overline{\zeta},\overline{\eta})$, of degrees $(-1,-1,0,0)$ and $(-2,0,0,0)$, respectively.

Also, we have the relations, valid for any $(\zeta,\eta)\in L(W)$:

(i) $\phi(\beta)(\zeta,\eta)=-\phi(\beta)(\eta,\zeta)$;

(ii) $\phi(\beta)(\zeta,\eta)$ *is holomorphic in the variables* ζ *and* η;

(iii) $\psi(\beta)(\zeta,\eta)$ *is holomorphic in the variable* η;

(iv) $\partial_{\overline{\zeta}}\psi(\beta)(\zeta,\eta)=B(\zeta)$;

(v) $\partial_\eta\psi(\beta)(\zeta,\eta)=\partial_\zeta\phi(\beta)(\zeta,\eta)$.

Finally, the form β is exact if and only if $\phi(\beta)$ vanishes, if and only if there exists a unique smooth function $\alpha(\beta)$, defined globally on W, taking values in $\Theta(-2)$, such that $\overline{\partial}\alpha(\beta)=\beta$ and if the function $\alpha(\beta)$ is represented by a smooth function, $\alpha(\beta)(\zeta)$, homogeneous of degrees $(-2,0)$ in the variables $(\zeta,\overline{\zeta})$, defined on $H(W)$, then we have the relation, valid for all $(\zeta,\eta)\in L(W)$:

(vi) $\alpha(\beta)(z)=\psi(\beta)(\zeta,\eta)$.

Proof. Equation (7.48) is a rescaling by a factor of $(-2\mathrm{i}\pi)^{-1}$ of equation (7.10).

Equation (7.49) is simply a rewrite (and rescaling) of the integrals of equations (7.22) and (7.23), using polar coordinates for each integral and combining them to give the quantity $\psi(\zeta,\eta)$ according to equation (7.20).

Property (i) is equation (7.11). Property (ii) is lemma 7.3. Property (iii) is theorem 7.8. Property (iv) is corollary 7.9. Property (v) is lemma 7.16.

If the function $\phi(\beta)$ vanishes, then the $\overline{\partial}$-exactness of β and the existence of the function $\alpha(\beta)$, obeying property (vi) are established in corollary 7.17. Conversely, if the form β is $\overline{\partial}$-exact, then by lemma 7.1, the function $\phi(\beta)$ vanishes. Also if the function $\alpha(\beta)$ exists obeying property (vi), then by property (iv), the form β is $\overline{\partial}$-exact and $\overline{\partial}\alpha(\beta)=\beta$. ∎

Next we wish to consider the analogue of theorem 7.18 for the case of the domain $P_{\mathbb{C}}-P$. Since this domain is not admissible, we have to do a little more work.

Corollary 7.19. *Let β be a smooth $\overline{\partial}$-closed (0, 1) form with values in the sheaf $\Theta(-2)$, defined globally on the space $P_{\mathbb{C}}-P$. So the form β is represented by a smooth function, $B(\upsilon)$, which takes values in the space $V^*_{\mathbb{C}}$, is defined globally on the space H and homogeneous of degrees $(-2,-1)$ in the variables $(\upsilon,\overline{\upsilon})$ and which obeys the equations: $B(\upsilon)\cdot\overline{\upsilon}=0$ and $\partial_{\overline{\upsilon}}\wedge B(\upsilon)=0$.*

Define the smooth function $\psi(\beta)$ on the space $L(P_{\mathbb{C}}-P)$, by the formula, valid for any $(\zeta,\eta)\in L(P_{\mathbb{C}}-P)$:

$$\psi(\beta)(\zeta,\eta)\equiv-\pi^{-1}\int_{-\pi}^{\pi}\int_{0}^{1}[B(\zeta+r\mathrm{e}^{\mathrm{i}\theta}\eta)+rB(r\zeta+\mathrm{e}^{\mathrm{i}\theta}\eta)]\cdot\overline{\eta}\mathrm{e}^{-\mathrm{i}\theta}\,\mathrm{d}r\,\mathrm{d}\theta. \qquad (7.50)$$

Then we have the relations, valid for any $(\zeta,\eta)\in L(P_{\mathbb{C}}-P)$:

(i) $\partial_{\overline{\zeta}}\psi(\beta)(\zeta,\eta)=B(\zeta)$;

(ii) $\partial_{\eta}\psi(\beta)(\zeta,\eta)=\partial_{\overline{\eta}}\psi(\beta)(\zeta,\eta)=0$.

Also, there exists a unique smooth function $\alpha(\beta)$, defined globally on $P_{\mathbb{C}}-P$, taking values in $\Theta(-2)$, such that $\overline{\partial}(\alpha(\beta))=\beta$ and if $\alpha(\beta)$ is represented by a smooth function, $\alpha(\beta)(\zeta,\overline{\zeta})$, homogeneous of degrees $(-2,0)$ in the pair of variables $(\zeta,\overline{\zeta})$, defined on H, then we have the relation, valid for all $(\zeta,\eta)\in L(P_{\mathbb{C}}-P)$:

(iii) $\alpha(\beta)(\zeta)=\psi(\beta)(\zeta,\eta)$.

Proof. All the definitions and results of this section apply to this domain, up to and including lemma 7.16. Let β be a smooth $\overline{\partial}$-closed $(0,1)$ form with values in $\Theta(-2)$, defined globally on $P_{\mathbb{C}}-P$. By corollary 6.7, the form β is $\overline{\partial}$-exact and a unique smooth function α exists, defined on $P_{\mathbb{C}}-P$, taking values in $\Theta(-2)$, such that $\overline{\partial}\alpha=\beta$.

Let W be an admissible open subset of $P_{\mathbb{C}}-P$. Then theorem 7.18 applies to the restriction of β to W. By the uniqueness of the quantity $\alpha(\beta)$ of theorem 7.18, the function $\alpha(\beta)$ must be just the restriction of the function α to W. Since every point and every projective line in $P_{\mathbb{C}}-P$ belong to some admissible open subset W of the space $P_{\mathbb{C}}-P$, we may read off the requisite formulas directly from theorem 7.18. ∎

8. The forward direction: construction of $\Phi(f)$ given f

Let a function $f\in C^{\infty}(V,-2)$ be given. Multiply f by the V-valued function z. The resulting V-valued function zf is homogeneous of degree -1. We regard the quantity

zf as taking values in the space Ω^1. Fix $x \in M$ and consider $x^*(zf)$, the restriction of the function zf to the space x'. We apply the invariant Hilbert transform, H_x, of § 5 to the function $x \wedge (zf)$. By theorem 5.15 there are unique $\Omega^1_{\mathbb{C}}$-valued functions $f_\pm(x,z)$, globally defined on the spaces $S^\pm$, fibre holomorphic and homogeneous of degree -1 in the variable z, such that these functions extend smoothly to the domains $S_\pm \cup S_0$ and obey the relations, for all $(x,v) \in S$:

$$f_+(x,v) - f_-(x,v) = x^*(zf)(v), \tag{8.1}$$

$$f_+(x,v) + f_-(x,v) = -\mathrm{i}H_x(x^*(zf))(v). \tag{8.2}$$

By equation (8.1), we have: $v \wedge f_+(x,v) = v \wedge f_-(x,v)$, for each $(x,v) \in S$, since $v \wedge x^*(zf)(v) = x^*((z \wedge z)f)(v) = 0$. Therefore, there exists a smooth function $g(x,\zeta)$, globally defined on the space $S_{\mathbb{C}}$, taking values in $\Omega^2_{\mathbb{C}}$, fibre holomorphic and homogeneous of degree zero, such that the restrictions of the function g to the domains $S_\pm \cup S_0$ agree with each of the quantities $\zeta \wedge f_\pm(x,z)$ on their respective domains. By Liouville's theorem, for each fixed $x \in M$, the function $g(x,z)$ must be independent of the variable $z \in x'_{\mathbb{C}}$. So the function g is the pullback to the space $S_{\mathbb{C}}$ of a smooth function, h, defined on the space M, i.e. we have $h(x) = g(x,\zeta)$, for all $(x,\zeta) \in S_{\mathbb{C}}$. By equation (8.2), we have the equation: $2h(x) = -\mathrm{i}v \wedge H_x(x^*(zf))(v)$, for each $(x,v) \in S$. In particular, the quantity $v \wedge H_x(x^*(zf))(v)$ is independent of $v \in x'$, for any $x \in M$; since it is real it represents an element of Ω^2. As $x \in M$ varies, we obtain a smooth function $\Phi(f)$, globally defined on the space M and taking values in Ω^2, such that for any $(x,v) \in S$, we have the relation $\Phi(f)(x) = v \wedge H_x(x^*(zf))(v)$. Note that we have $v \wedge \Phi(f)(x) = 0$, whenever $v \in x \in M$.

We have proved:

Lemma 8.1. *Given any $f \in C^\infty(V,-2)$, there exists a function*

$$\Phi(f) \in C^\infty(M, \Omega^2),$$

such that, for any $(x,v) \in S$:

$$\Phi(f)(x) = v \wedge H_x(x^*(zf))(v). \tag{8.3}$$

The function $\Phi(f)$ is given in terms of the functions $f_\pm$ by the equations, valid for all $(x,z) \in S_\pm \cup S_0$:

$$\Phi(f)(x) = 2\mathrm{i}\zeta \wedge f_\pm(x,\zeta). \tag{8.4}$$

Furthermore, we have $v \wedge \Phi(f)(x) = 0$, for any $v \in x \in M$.

We can derive an explicit formula for the function $\Phi(f)$ as follows.

First, by lemma 5.3, there is a unique function $h(f) \in C^\infty((x')^2, \mathrm{End}(x))$, such that, for all $(u,v) \in (x')^2$:

$$h(f)(u,v)(u) = -vf(v), \qquad h(f)(u,v)(v) = -uf(u). \tag{8.5}$$

By lemma 5.4, the one form $h(f)(u,v)(\mathrm{d}u)$ is closed, for each fixed $v \in x'$. By definition 5.6, we have the formula: $H_x(x^*(zf))(v) = (2\pi)^{-1}\int_x [h(f)(u,v)(\mathrm{d}u)]$. Therefore, by equation (8.3), we obtain the explicit formula, valid for any $v \in x'$ and any $x \in M$:

$$\Phi(f)(x) = (2\pi)^{-1}\int_x [v \wedge h(f)(u,v)(\mathrm{d}u)]. \tag{8.6}$$

Now consider the quantity $v \wedge h(f)(u,v) - f(u)(u \wedge \delta)$. If we act on the vector u, then, using equation (8.5), we find

$$[v \wedge h(f)(u,v) - f(u)u \wedge \delta](u) = -(v \wedge v)f(v) - f(u)(u \wedge u) = 0. \tag{8.7}$$

If we act on the vector v, then, using equation (8.5), we get

$$[v \wedge h(f)(u,v) - f(u)u \wedge \delta](v) = -(v \wedge u)f(u) - f(u)(u \wedge v) = 0. \tag{8.8}$$

By equations (8.7) and (8.8), we have $v \wedge h(f)(u,v) - f(u)(u \wedge \delta) = 0$, whenever the vectors u and v are linearly independent, so whenever the quantity $u \wedge v$ is non-zero. So, by continuity, we have, for all $(u,v) \in (x')^2$,

$$v \wedge h(f)(u,v) = f(u)(u \wedge \delta). \tag{8.9}$$

Substituting equation (8.9) into equation (8.6), we get

$$\Phi(f)(x) = (2\pi)^{-1} \int_x [f(u)u \wedge \mathrm{d}u)]. \tag{8.10}$$

Comparing equations (8.10) and (4.1), we have exact agreement. So we have proved:

Lemma 8.2. *The function $\Phi(f)$ defined by equation (8.3) agrees with the Radon transform of the function f, as given by definition 4.3.*

Now the derivatives $F_{\mathbb{C}}(f_{\pm})$ take values in $V^*_{\mathbb{C}} \otimes V_{\mathbb{C}} \otimes \Omega'(\mathbb{C})$, where $F_{\mathbb{C}}$ is the operator $E - \delta_{\mathbb{C}}$. After identifying $V_{\mathbb{C}}$ with $\Omega^1_{\mathbb{C}}$, we may regard $F_{\mathbb{C}}(g_{\pm})$ as having values in $V^*_{\mathbb{C}} \otimes V_{\mathbb{C}} \otimes V_{\mathbb{C}}$. Then skew symmetrization gives quantities $\wedge_{\mathbb{C}}(F_{\mathbb{C}}(f_{\pm}))$ taking values in $\Omega^1(V^*_{\mathbb{C}}) \otimes \Omega^2_{\mathbb{C}}$.

Lemma 8.3. *The functions $f_{\pm}$, restricted to the space S, obey the differential equations: $\wedge_{\mathbb{C}}(F_{\mathbb{C}}(f_{\pm})) = 0$.*

Proof. Apply the operator $F_{\mathbb{C}}$ to both sides of equation (8.1) and take the skew part. We find the relation $\wedge_{\mathbb{C}}(F_{\mathbb{C}}(x^*(zf))) = x^*(\wedge(F(zf))) = 0$, since we have: $\wedge(F(zf)) = \wedge(F(z))f + (F(f)) \wedge z = (E(f)) \wedge z = (\partial f) \otimes z \wedge z = 0$, using the facts that $\delta(f) = 0$, so $F(f) = E(f) = (\partial f) \otimes z$ and $E \otimes (z) = z \otimes \partial \otimes (z) = z \otimes \delta = \delta(z)$, so $F(z) = 0$.

This gives the formula, valid on the space S: $\wedge_{\mathbb{C}}(F_{\mathbb{C}}(f_+)) = \wedge_{\mathbb{C}}(F_{\mathbb{C}}(f_-))$. But each of the quantities $\wedge_{\mathbb{C}}(F_{\mathbb{C}}(f_{\pm}))$ is fibre holomorphic and homogeneous of degree minus one, on its domain. Then the relation $\wedge_{\mathbb{C}}(F_{\mathbb{C}}(f_+)) = \wedge_{\mathbb{C}}(F_{\mathbb{C}}(f_-))$ on the space S, shows that the quantities $\wedge_{\mathbb{C}}(F_{\mathbb{C}}(f_{\pm}))(x,z)$, patch together to give a global fibre holomorphic function on $S_{\mathbb{C}}$, homogeneous of degree minus one, which vanishes by Liouville's theorem, as required for the lemma. ■

Switching to abstract indices, equation (8.1) reads

$$v^\alpha f(x,v) = f^\alpha_+(x,v) - f^\alpha_-(x,v), \tag{8.11}$$

valid for all $(x,v) \in S$.

The result of lemma 8.3 then reads

$$E^{[\beta}_\gamma f^{\alpha]}_\pm + \delta^{[\beta}_\gamma f^{\alpha]}_\pm = 0. \tag{8.12}$$

Accordingly, we have the relations, for unique symmetric (complex) tensor fields $f^{\alpha\beta}_{\pm\gamma}$,

$$E^{\beta}_{\gamma} f^{\alpha}_{\pm} = \delta^{\alpha}_{\gamma} f^{\beta}_{\pm} + f^{\alpha\beta}_{\pm\gamma}. \tag{8.13}$$

Applying the operator E^{δ}_{ϵ} to equation (8.13) and taking the commutator, using the fact that the derivative operator E^{α}_{β} obeys the commutation relation: $E^{\delta}_{\epsilon}E^{\beta}_{\gamma} - E^{\beta}_{\gamma}E^{\delta}_{\epsilon} = \delta^{\beta}_{\epsilon}E^{\delta}_{\gamma} - \delta^{\delta}_{\gamma}E^{\beta}_{\epsilon}$, we obtain the following identity:

$$E^{\delta}_{\epsilon} f^{\alpha\beta}_{\pm\gamma} - E^{\beta}_{\gamma} f^{\alpha\delta}_{\pm\epsilon} = \delta^{\alpha}_{\epsilon} f^{\beta\delta}_{\pm\gamma} - \delta^{\alpha}_{\gamma} f^{\beta\delta}_{\pm\epsilon} + \delta^{\beta}_{\epsilon} f^{\alpha\delta}_{\pm\gamma} - \delta^{\delta}_{\gamma} f^{\alpha\beta}_{\pm\epsilon}. \tag{8.14}$$

Equation (8.14) may be solved directly for the derivatives $E^{\delta}_{\epsilon} f^{\alpha\beta}_{\pm\gamma}$:

$$E^{\delta}_{\epsilon} f^{\alpha\beta}_{\pm\gamma} = \delta^{\alpha}_{\epsilon} f^{\beta\delta}_{\pm\gamma} + \delta^{\beta}_{\epsilon} f^{\alpha\delta}_{\pm\gamma} + f^{\alpha\beta\delta}_{\pm\gamma\epsilon}. \tag{8.15}$$

In equation (8.15), the tensors $f^{\alpha\beta\delta}_{\pm\gamma\epsilon}$ are totally symmetric. Combining equations (8.13) and (8.15), we find the following formula for the second derivatives of the quantities $f^{\alpha}_{\pm}$:

$$E^{\delta}_{\epsilon}E^{\beta}_{\gamma} f^{\alpha}_{\pm} = \delta^{\alpha}_{\gamma}E^{\delta}_{\epsilon} f^{\beta}_{\pm} + E^{\delta}_{\epsilon} f^{\alpha\beta}_{\pm\gamma} = \delta^{\alpha}_{\gamma}\delta^{\beta}_{\epsilon} f^{\delta}_{\pm} + \delta^{\alpha}_{\gamma} f^{\beta\delta}_{\pm\epsilon} + \delta^{\alpha}_{\epsilon} f^{\beta\delta}_{\pm\gamma} + \delta^{\beta}_{\epsilon} f^{\alpha\delta}_{\pm\gamma} + f^{\alpha\beta\delta}_{\pm\gamma\epsilon}. \tag{8.16}$$

Skew symmetrizing equation (8.16) and using equation (8.13) again, we get the simple relation:

$$E^{[\delta}_{[\epsilon}E^{\beta]}_{\gamma]} f^{\alpha}_{\pm} = \delta^{[\beta}_{[\epsilon}E^{\delta]}_{\gamma]} f^{\alpha}_{\pm}. \tag{8.17}$$

Using these relations, we have the following second derivative formula:

$$\begin{aligned}
E^{[\delta}_{[\epsilon}E^{\beta]}_{\gamma]}(\zeta^{[\zeta} f^{\alpha]}_{\pm}) &= \delta^{[\beta}_{[\epsilon}E^{\delta]}_{\gamma]}(\zeta^{[\zeta} f^{\alpha]}_{\pm}) - (\delta^{[\beta}_{[\epsilon}E^{\delta]}_{\gamma]}\zeta^{[\zeta})f^{\alpha]}_{\pm} + (E^{[\delta}_{[\epsilon}E^{\beta]}_{\gamma]}\zeta^{[\zeta})f^{\alpha]}_{\pm} + 2\zeta^{[\beta}E^{\delta]}_{[\epsilon}\delta^{[\zeta}_{\gamma]}f^{\alpha]}_{\pm} \\
&= \delta^{[\beta}_{[\epsilon}E^{\delta]}_{\gamma]}(\zeta^{[\zeta} f^{\alpha]}_{\pm}) - \delta^{[\beta}_{[\epsilon}\zeta^{\delta]}\delta^{[\zeta}_{\gamma]}f^{\alpha]}_{\pm} - \delta^{[\delta}_{[\epsilon}\zeta^{\beta]}\delta^{[\zeta}_{\gamma]}f^{\alpha]}_{\pm} + 2\zeta^{[\beta}E^{\delta]}_{[\epsilon}\delta^{[\zeta}_{\gamma]}f^{\alpha]}_{\pm} \\
&= \delta^{[\beta}_{[\epsilon}E^{\delta]}_{\gamma]}(\zeta^{[\zeta} f^{\alpha]}_{\pm}) + 2\zeta^{[\beta}E^{\delta]}_{[\epsilon}\delta^{[\zeta}_{\gamma]}f^{\alpha]}_{\pm} \\
&= \delta^{[\beta}_{[\epsilon}E^{\delta]}_{\gamma]}(\zeta^{[\zeta} f^{\alpha]}_{\pm}) + 2E^{[\alpha}_{[\epsilon}\delta^{\zeta]}_{\gamma]}f^{[\delta}_{\pm})\zeta^{\beta]} + 2\zeta^{[\beta}f^{\delta]}_{\pm}\delta^{[\alpha}_{[\epsilon}\delta^{\zeta]}_{\gamma]} - 2f^{[\alpha}_{\pm}\delta^{\zeta]}_{[\gamma}\delta^{[\delta}_{\epsilon]}\zeta^{\beta]} \\
&= \delta^{[\beta}_{[\epsilon}E^{\delta]}_{\gamma]}(\zeta^{[\zeta} f^{\alpha]}_{\pm}) + 2E^{[\alpha}_{[\epsilon}\delta^{\zeta]}_{\gamma]}(f^{[\delta}_{\pm}\zeta^{\beta]}) - 2\zeta^{[\alpha}\delta^{\zeta]}_{[\gamma}(f^{[\delta}_{\pm}\delta^{\beta]}_{\epsilon]}) \\
&\quad + 2\zeta^{[\beta}f^{\delta]}_{\pm}\delta^{[\alpha}_{[\epsilon}\delta^{\zeta]}_{\gamma]} - 2f^{[\alpha}_{\pm}\delta^{\zeta]}_{[\gamma}\delta^{[\delta}_{\epsilon]}\zeta^{\beta]} \\
&= \delta^{[\beta}_{[\epsilon}E^{\delta]}_{\gamma]}(\zeta^{[\zeta} f^{\alpha]}_{\pm}) + 2E^{[\alpha}_{[\epsilon}\delta^{\zeta]}_{\gamma]}(f^{[\delta}_{\pm}\zeta^{\beta]}) + 2\delta^{[\alpha}_{[\epsilon}\delta^{\zeta]}_{\gamma]}\zeta^{[\beta}f^{\delta]}_{\pm} \\
&\quad + \tfrac{1}{2}(-f^{\alpha}_{\pm}\delta^{\zeta}_{[\gamma}\delta^{\delta}_{\epsilon]}\zeta^{\beta} + f^{\zeta}_{\pm}\delta^{\alpha}_{[\gamma}\delta^{\delta}_{\epsilon]}\zeta^{\beta} - \zeta^{\alpha}\delta^{\zeta}_{[\gamma}f^{\delta}_{\pm}\delta^{\beta}_{\epsilon]} + \zeta^{\zeta}\delta^{\alpha}_{[\gamma}f^{\delta}_{\pm}\delta^{\beta}_{\epsilon]}) \\
&\quad - \tfrac{1}{2}(-f^{\alpha}_{\pm}\delta^{\zeta}_{[\gamma}\delta^{\beta}_{\epsilon]}\zeta^{\delta} + f^{\zeta}_{\pm}\delta^{\alpha}_{[\gamma}\delta^{\beta}_{\epsilon]}\zeta^{\delta} - \zeta^{\alpha}\delta^{\zeta}_{[\gamma}f^{\beta}_{\pm}\delta^{\delta}_{\epsilon]} + \zeta^{\zeta}\delta^{\alpha}_{[\gamma}f^{\beta}_{\pm}\delta^{\delta}_{\epsilon]}) \\
&= \delta^{[\beta}_{[\gamma}E^{\delta}_{\epsilon]}](\zeta^{[\alpha} f^{\zeta]}_{\pm}) - 2E^{[\alpha}_{[\gamma}\delta^{\zeta]}_{\epsilon]}(\zeta^{[\beta}f^{\delta]}_{\pm}) - 2\delta^{\alpha}_{[\gamma}\delta^{\zeta}_{\epsilon]}\zeta^{[\beta}f^{\delta]}_{\pm} \\
&\quad - \delta^{\delta}_{[\gamma}\delta^{\zeta}_{\epsilon]}\zeta^{[\alpha}f^{\beta]}_{\pm} + \delta^{\alpha}_{[\gamma}\delta^{\delta}_{\epsilon]}\zeta^{[\beta}f^{\zeta]}_{\pm} + \delta^{\beta}_{[\gamma}\delta^{\zeta}_{\epsilon]}\zeta^{[\alpha}f^{\delta]}_{\pm} - \delta^{\alpha}_{[\gamma}\delta^{\beta}_{\epsilon]}\zeta^{[\delta}f^{\zeta]}_{\pm}.
\end{aligned} \tag{8.18}$$

Using equation (8.4), we may rewrite equation (8.18) in terms of the field $\Phi(f)$, as follows:

$$\begin{aligned}
0 &= E^{[\delta}_{[\epsilon}E^{\beta]}_{\gamma]}\Phi(f)^{\zeta\alpha} - \delta^{[\beta}_{[\gamma}E^{\delta]}_{\epsilon]}\Phi(f)^{\alpha\zeta} + 2\delta^{[\alpha}_{[\gamma}E^{\zeta]}_{\epsilon]}\Phi(f)^{\beta\delta} + 2\delta^{\alpha}_{[\gamma}\delta^{\zeta}_{\epsilon]}\Phi(f)^{\beta\delta} \\
&\quad + \delta^{\zeta}_{[\gamma}\delta^{\delta}_{\epsilon]}\Phi(f)^{\beta\alpha} - \delta^{\alpha}_{[\gamma}\delta^{\delta}_{\epsilon]}\Phi(f)^{\beta\zeta} - \delta^{\zeta}_{[\gamma}\delta^{\beta}_{\epsilon]}\Phi(f)^{\delta\alpha} + \delta^{\alpha}_{[\gamma}\delta^{\beta}_{\epsilon]}\Phi(f)^{\delta\zeta} \\
&= E^{[\delta}_{[\epsilon}E^{\beta]}_{\gamma]}\Phi(f)^{\zeta\alpha} + \delta^{[\delta}_{[\epsilon}E^{\beta]}_{\gamma]}\Phi(f)^{\zeta\alpha} + 2\delta^{[\alpha}_{[\gamma}E^{\zeta]}_{\epsilon]}\Phi(f)^{\beta\delta} \\
&\quad + 2\delta^{\alpha}_{[\gamma}\delta^{\zeta}_{\epsilon]}\Phi(f)^{\beta\delta} + 4\delta^{[\beta}_{[\gamma}\Phi(f)^{\delta][\alpha}\delta^{\zeta]}_{\epsilon]}.
\end{aligned} \tag{8.19}$$

Also, using equation (8.12), we have the first derivative equation:

$$E_\gamma^{[\beta}\zeta^\delta f_\pm^{\alpha]} = -2\delta_\gamma^{[\beta}\zeta^\delta f_\pm^{\alpha]}. \tag{8.20}$$

Writing equation (8.20) in terms of the quantity $\Phi(f)^{\alpha\beta}$, we have the equation:

$$E_\delta^{[\alpha}\Phi(f)^{\beta\gamma]} + 2\delta_\delta^{[\alpha}\Phi(f)^{\beta\gamma]} = 0. \tag{8.21}$$

Using equation (8.21), we may rewrite equation (8.19) as follows:

$$\begin{aligned}
0 &= E_{[\epsilon}^{[\delta}E_{\gamma]}^{\beta]}\Phi(f)^{\zeta\alpha} + \delta_{[\epsilon}^{[\delta}E_{\gamma]}^{\beta]}\Phi(f)^{\zeta\alpha} + 4E_{[\epsilon}^{[\delta}\Phi(f)^{\beta][\zeta}\delta_{\gamma]}^{\alpha]} \\
&\quad + 8\delta_{[\epsilon}^{[\delta}\Phi(f)^{\beta][\zeta}\delta_{\gamma]}^{\alpha]} - 2\delta_{[\gamma}^{\alpha}\delta_{\epsilon]}^{\zeta}\Phi(f)^{\beta\delta} + 4\delta_{[\gamma}^{[\beta}\Phi(f)^{\delta][\alpha}\delta_{\epsilon]}^{\zeta]} \\
&= E_{[\epsilon}^{[\delta}E_{\gamma]}^{\beta]}\Phi(f)^{\zeta\alpha} + 4E_{[\epsilon}^{[\delta}\Phi(f)^{\beta][\zeta}\delta_{\gamma]}^{\alpha]} + \delta_{[\epsilon}^{[\delta}E_{\gamma]}^{\beta]}\Phi(f)^{\zeta\alpha} \\
&\quad + 4\delta_{[\epsilon}^{[\delta}\Phi(f)^{\beta][\zeta}\delta_{\gamma]}^{\alpha]} + 2\delta_{[\epsilon}^{[\zeta}\delta_{\gamma]}^{\alpha]}\Phi(f)^{\delta\beta}.
\end{aligned} \tag{8.22}$$

Comparing equations (8.21) and (8.22) with equations (4.8) and (4.9) above, we have exact agreement.

We have proved:

Lemma 8.4. *The field $\Phi(f)$ belongs to the space $Z(M)$.*

Now suppose that the field $\Phi(f)$ is identically zero. Then we have the formulas $\zeta\wedge f_\pm(x,\zeta) = 0$, which immediately gives the relations $f_\pm(x,z) = \zeta k_\pm(x,\zeta)$, for some functions $k_\pm$, which are defined for all $(x,\zeta)\in S_\pm\cup S_0$ and are fibre holomorphic and homogeneous of degree minus two. Equation (8.12) gives the following relation:

$$\wedge_{\mathbb{C}}[\zeta\otimes E(k_\pm)] = 0. \tag{8.23}$$

Next, pull back the functions $k_\pm$ (restricted to the domains $S_\pm$) to the space H, along the maps $s_\pm$: so put $k_\pm s_\pm \equiv g_\pm$. Then each of the functions $g_\pm$ is globally defined on the space H. On the space H, the derivatives $E_\gamma^\beta g_\pm$ may be expressed as $E(g_\pm) = \zeta\otimes\partial_\zeta g_\pm + \bar\zeta\otimes\partial_{\bar\zeta}g_\pm$. If $\zeta\in H$, we have $\zeta\wedge\bar\zeta\neq 0$, so equation (8.23) gives the simple formula:

$$\partial_{\bar\zeta}g_\pm = 0. \tag{8.24}$$

Thus the quantities $g_\pm$ are holomorphic on H. Since they are homogeneous of degree minus two, they represent global holomorphic sections of the sheaf $\Theta(-2)$ over the space $P_{\mathbb{C}} - P$, which is the projective image, $p_{\mathbb{C}}(H)$ of the space H. For any y in H there is a projective line through $p_{\mathbb{C}}(y)$ lying entirely in the space $P_{\mathbb{C}} - P$. Since the functions $g_\pm$, restricted to any such line represent holomorphic globally defined sections of $\Theta(-2)$ and since the only such section vanishes identically, by Liouville's theorem, we must have $g_\pm$ vanishing on the line. In particular, we have $g_\pm(y) = 0$. Since y is arbitrary in H, the functions $g_\pm$ must vanish identically. Hence the functions $k_\pm$ and $f_\pm$ must vanish identically. By equation (8.1), the function f must also vanish identically. We have proved:

Theorem 8.5. *The Radon transform Φ is an injection.*

9. The backward direction: construction of the function f given its Radon transform $\Phi(f)$

Given $f \in C^\infty(V, -2)$, we may obtain its Radon transform $\Phi(f) \in Z(M)$. The aim of this section is to recover the function f, from the function $\Phi(f)$. By theorem 8.5, if $\Phi(f)$ is known, then f is uniquely determined.

From § 8, we know that unique $V_{\mathbb{C}}$-valued functions $f_\pm(x, \zeta)$ exist, homogeneous of degrees $(-1, 0)$ in the variables $(\zeta, \bar{\zeta})$, defined and fibre holomorphic on the domains $S_\pm \cup S_0$ and such that we have the relations:

$$f_+(x, v) - f_-(x, v) = v f(v), \tag{9.1}$$

valid for all $v \in x'$ and all $x \in M$,

$$\Phi(f)(x) = 2\mathrm{i}\zeta \wedge f_\pm(x, \zeta), \tag{9.2}$$

valid for all $(x, \zeta) \in S_\pm \cup S_0$.

Also, by corollary 5.16, we have $\overline{f_-(x, \zeta)} = -f_+(x, \bar{\zeta})$, for all $(x, \zeta) \in S_- \cup S_0$. This relation in particular entails that for all $x \in M$ and all $v \in x'$, we have $f_-(x, v) = -\overline{f_+(x, v)}$. Hence either of the functions $f_\pm$ determines the other completely and from equation (9.1), we have the following formulas, valid for all $v \in x'$ and all $x \in M$:

$$v f(v) = \pm 2\,\mathrm{Re}(f_\pm(x, v)). \tag{9.3}$$

Recall that $X \wedge v = 0$, for any $v \in m(X)$ and any $X \in N$. Then equation (9.1) gives immediately the following relation: $X \wedge f_+(m(X), v) = X \wedge f_-(m(X), v)$, for any $X \in N$ and any $v \in m(X)'$. This relation, combined with the homogeneity of the functions $f_\pm$, shows in particular that the pair of functions $X \wedge f_\pm(m(X), \zeta)$, for each fixed $X \in N$, patch together to give a single global holomorphic function defined for all $\zeta \in m(X)'_{\mathbb{C}}$, which is homogeneous of degree minus one in the variable ζ. By Liouville's theorem such a function must be zero. So we have the key relations, valid for all (X, ζ), such that $(m(X), \zeta) \in S_\pm \cup S_0$:

$$X \wedge f_\pm(m(X), \zeta) = 0. \tag{9.4}$$

Now we may pull back the functions $f_\pm$ restricted to the domains $S_\pm$ along the maps $s_\pm$ to give $V_{\mathbb{C}}$-valued functions, $g_\pm(\zeta)$, defined for all $\zeta \in H$. (So we have $g_\pm \equiv f_\pm s_\pm$.) Recall that if $\zeta \in H$, then $s_\pm(\zeta) = (x_\pm(\zeta), \zeta)$, where we have $x_\pm(\zeta) = m(X_\pm(\zeta))$ and $X_\pm(\zeta) = \pm 2\zeta^+ \wedge \zeta^- = \pm \mathrm{i}\zeta \wedge \bar{\zeta}$. Then equation (9.4) may be rewritten in terms of the functions $g_\pm$ as follows, valid for all $\zeta \in H$:

$$\zeta \wedge \bar{\zeta} \wedge g_\pm(\zeta) = 0. \tag{9.5}$$

Also equation (9.3) may be rewritten directly in terms of the functions $g_\pm$ as follows; for any fixed $v \in V'$, we have

$$v f(v) = \pm 2 \lim_{\zeta \to v; \zeta \in H, v \wedge \zeta \wedge \bar{\zeta} = 0} \mathrm{Re}(g_\pm(\zeta)) = \pm 2 \lim_{t \to 0} \mathrm{Re}(g_\pm(v + \mathrm{i}tw)). \tag{9.6}$$

In the second limit, the variable t is real and the vector $w \in V'$ must be linearly independent of the vector v.

From equation (9.5), since by definition $\zeta \wedge \bar{\zeta} \neq 0$, whenever $\zeta \in H$, we may decompose the functions $g_\pm$ as follows:

$$g_\pm(\zeta) = \zeta \chi_\pm(\zeta) + \bar{\zeta} \phi_\pm(\zeta). \tag{9.7}$$

In equation (9.7), the smooth scalar functions $\chi_\pm$ and $\phi_\pm$ are globally defined on the space H and are uniquely determined by the functions $g_\pm$. Note that none of the functions $\chi_\pm$ nor $\phi_\pm$, if regarded as functions on the spaces $S_\pm$ (via the maps $s_\pm$), can be expected to have a smooth extension to the space $S_\pm \cup S_0$. Also the functions $\chi_\pm$ have homogeneity $(-2,0)$, whereas the functions $\phi_\pm$ have homogeneity $(-1,-1)$, in the variables $(\zeta, \overline{\zeta})$.

Combining equations (9.2) and (9.7), we find the relations, valid for all $\zeta \in H$:

$$2\mathrm{i}\zeta \wedge \overline{\zeta}\phi_\pm(\zeta) = \Phi(f)(x_\pm(\zeta)). \tag{9.8}$$

By equation (9.7) the functions $\phi_\pm$ are directly determined from the given field $\Phi(f)$. So if a method is found to determine the functions $\chi_\pm$ directly from the field $\Phi(f)$, then the functions $g_\pm$ will be completely known from equation (9.7). Via the maps $s_\pm$, the functions $f_\pm$ will be known on the domains $S_\pm$. By continuity the extensions of the functions $f_\pm$ to their full domains $S_\pm \cup S_0$ will be known. In particular, the quantities $f_\pm(x,v)$ will be known for any $(x,v) \in S$ and then the function f will be recovered from equation (9.1). Equivalently, once the functions $g_\pm$ have been determined, we may recover the function f by using equation (9.6).

Note that from equation (9.8), since the quantity $\zeta \wedge \overline{\zeta}$ is killed by the differential operator $\partial_\zeta \cdot \overline{\zeta}$, we obtain the relations:

$$(\partial_\zeta \phi_\pm) \cdot \overline{\zeta} = 0. \tag{9.9}$$

Lemma 9.1. *Let* $\beta_\pm(\zeta) \equiv (\partial_\zeta \phi_\pm(\zeta)) \cdot \overline{\mathrm{d}\zeta}$, *for any* $\zeta \in H$. *Then the* $(0,1)$ *forms* $\beta_\pm$ *are smooth,* $\overline{\partial}$*-closed and globally defined on the space* H. *Further they represent smooth* $\overline{\partial}$*-closed* $(0,1)$ *forms on* $P_{\mathbb{C}} - P$, *with values in the sheaf* $\Theta(-2)$.

Proof. Since $\Phi(f) \in Z(M)$, the forms $\beta_\pm$ are $\overline{\partial}$-closed by equation (9.8) and lemma 4.9 and are clearly globally defined and smooth on the space H. The functions $\phi_\pm$ are homogeneous of degrees $(-1,-1)$ in the variables $(\zeta, \overline{\zeta})$ and since they also obey the equation (9.9), the forms $\beta_\pm$ are homogeneous of degrees $(-2,0)$ in the variables $(\zeta, \overline{\zeta})$, so represent $\overline{\partial}$-closed forms on the space $p_{\mathbb{C}}(H) = P_{\mathbb{C}} - P$ with coefficients in the sheaf $\Theta(-2)$. ∎

Lemma 8.3 connects the field $\Phi(f)$ and the functions $\chi_\pm$. From equation (8.12), we have

$$E_\gamma^{[\beta} f_\pm^{\alpha]} + \delta_\gamma^{[\beta} f_\pm^{\alpha]} = 0. \tag{9.10}$$

Pulling back equation (9.10) along the maps $s_\pm$ gives the following equation:

$$E_\gamma^{[\beta} g_\pm^{\alpha]} + \delta_\gamma^{[\beta} g_\pm^{\alpha]} = 0. \tag{9.11}$$

Using equation (9.7), we may rewrite equation (9.11) as follows:

$$0 = E_\gamma^{[\beta}(\zeta^{\alpha]}\chi_\pm + \overline{\zeta}^{\alpha]}\phi_\pm) + \delta_\gamma^{[\beta}(\zeta^{\alpha]}\chi_\pm + \overline{\zeta}^{\alpha]}\phi_\pm). \tag{9.12}$$

Acting on functions of the variable $\zeta \in H$, we have $E_\beta^\alpha = \zeta^\alpha \partial_\beta + \overline{\zeta}^\alpha \overline{\partial}_\beta$. Using this relation, equation (9.12) becomes

$$\begin{aligned}
0 &= \zeta^{[\beta}\partial_\gamma(\zeta^{\alpha]}\chi_\pm + \overline{\zeta}^{\alpha]}\phi_\pm) + \overline{\zeta}^{[\beta}\overline{\partial}_\gamma(\zeta^{\alpha]}\chi_\pm + \overline{\zeta}^{\alpha]}\phi_\pm) + \delta_\gamma^{[\beta}(\zeta^{\alpha]}\chi_\pm + \overline{\zeta}^{\alpha]}\phi_\pm) \\
&= \chi_\pm \zeta^{[\beta}\partial_\gamma \zeta^{\alpha]} + \overline{\zeta}^{[\alpha}\zeta^{\beta]}\partial_\gamma\phi_\pm + \overline{\zeta}^{[\beta}\zeta^{\alpha]}\overline{\partial}_\gamma\chi_\pm + \phi_\pm\overline{\zeta}^{[\beta}\overline{\partial}_\gamma\overline{\zeta}^{\alpha]} + \delta_\gamma^{[\beta}(\zeta^{\alpha]}\chi_\pm + \overline{\zeta}^{\alpha]}\phi_\pm) \\
&= \chi_\pm \zeta^{[\beta}\delta_\gamma^{\alpha]} + \overline{\zeta}^{[\alpha}\zeta^{\beta]}\partial_\gamma\phi_\pm + \overline{\zeta}^{[\beta}\zeta^{\alpha]}\overline{\partial}_\gamma\chi_\pm + \phi_\pm\overline{\zeta}^{[\beta}\delta_\gamma^{\alpha]} + \delta_\gamma^{[\beta}\zeta^{\alpha]}\chi_\pm + \delta_\gamma^{[\beta}\overline{\zeta}^{\alpha]}\phi_\pm \\
&= \overline{\zeta}^{[\alpha}\zeta^{\beta]}(\partial_\gamma\phi_\pm - \overline{\partial}_\gamma\chi_\pm).
\end{aligned} \tag{9.13}$$

Since $\zeta \wedge \overline{\zeta} \neq 0$, for any $\zeta \in H$, we deduce from equation (9.13) the following key equation for the quantities $\chi_{\pm}$:

$$\overline{\partial}_{\alpha}\chi_{\pm} = \partial_{\alpha}\phi_{\pm}. \tag{9.14}$$

We may summarize the above discussion as follows:

Theorem 9.2. *Let $f \in C^{\infty}(V, -2)$ be given and consider its Radon transform $\Phi(f) \in Z(M)$.*

Define the smooth real-valued functions $\phi_{\pm}$ on the space H, by the following formula, valid for any $z \in H$:

$$2\mathrm{i}\zeta \wedge \overline{\zeta}\phi_{\pm}(\zeta) \equiv \Phi(f)(x \pm (\zeta)). \tag{9.15}$$

Consider the following differential equations, for smooth complex-valued functions $\chi_{\pm}$, globally defined on the space H and homogeneous of degrees $(-2, 0)$ in the variables $(\zeta, \overline{\zeta})$.

$$\overline{\partial}_{\alpha}\chi_{\pm} = \partial_{\alpha}\phi_{\pm}. \tag{9.16}$$

Then the required solutions $\chi_{\pm}$ exist, are unique and are expressible directly in terms of certain integrals involving the functions $\phi_{\pm}$. Specifically we have the following formulas, valid for any $(\zeta, \eta) \in L(P_{\mathbb{C}} - P)$:

$$\chi_{\pm}(\zeta) \equiv -\pi^{-1}\int_{-\pi}^{\pi}\int_{0}^{1}\overline{\eta}^{\alpha}[\partial_{\alpha}\phi_{\pm}(\zeta + r\mathrm{e}^{\mathrm{i}\theta}\eta) + r\partial_{\alpha}\phi_{\pm}(r\zeta + \mathrm{e}^{\mathrm{i}\theta}\eta)]\mathrm{e}^{-\mathrm{i}\theta}\,\mathrm{d}r\,\mathrm{d}\theta. \tag{9.17}$$

Also let $V_{\mathbb{C}}$-valued functions $g_{\pm}$ be defined, globally on the space H, by the formulas, valid for any $\zeta \in H$:

$$g_{\pm}(\zeta) = \zeta\chi_{\pm}(\zeta) + \overline{\zeta}\phi_{\pm}(\zeta). \tag{9.18}$$

Next let functions $f_{\pm}$, taking values in $V_{\mathbb{C}}$, globally defined on the spaces $S_{\pm}$ be defined by the formulas:

$$g_{\pm} = f_{\pm}s_{\pm}. \tag{9.19}$$

Then the functions $f_{\pm}$ are fibre holomorphic and homogeneous of degree minus one in the variable ζ and possess smooth extensions (still called $f_{\pm}$) to the domains $S_{\pm} \cup S_0$. Finally, the function f may be recovered from the formula, valid for any $v \in x'$ and any $x \in M$:

$$f_{+}(x, v) - f_{-}(x, v) = vf(v). \tag{9.20}$$

Proof. We know that the functions $f_{\pm}$ and $g_{\pm}$ obeying equations (9.19) and (9.20) and the required holomorphy and homogeneity conditions exist and are unique. Further, the functions $g_{\pm}$ have a unique decomposition of the form of equation (9.18) and the components of the decomposition $\chi_{\pm}$ and $\phi_{\pm}$ obey the equations (9.15) and (9.16). Thus to prove the theorem, we just need to verify that equations (9.15) and (9.16) are by themselves sufficient to determine the functions $\chi_{\pm}$ and $\phi_{\pm}$ uniquely. For the functions $\phi_{\pm}$ this is immediate from equation (9.15). That the functions $\phi_{\pm}$ have the correct homogeneity also follows immediately from equation (9.15). Next, by lemma 9.1, the $(0, 1)$ forms $\beta_{\pm} \equiv (\partial_{\zeta}\phi_{\pm}) \cdot \mathrm{d}\overline{\zeta}$ represent $\overline{\partial}$-closed $(0, 1)$ forms with coefficients in the sheaf $\Theta(-2)$, globally defined on the space $P_{\mathbb{C}} - P$. By corollary 6.7, there exist unique sections $\chi_{\pm}$, of $\Theta(-2)$, globally defined on the

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space $P_{\mathbb{C}} - P$, such that $\overline{\partial}\chi_{\pm} = \beta_{\pm}$. Interpreted as functions on the space H, the quantities $\chi_{\pm}$ are smooth and globally defined, have the required homogeneity and obey the required equation, equation (9.16). Uniqueness of the functions $\chi_{\pm}$ on H is equivalent to their uniqueness on the space $P_{\mathbb{C}} - P$, so holds. So the functions $\chi_{\pm}$ and $\phi_{\pm}$ with the required properties exist and are indeed uniquely determined just from the equations (9.15) and (9.16). Finally, the integral expressions of equation (9.17), giving the functions $\chi_{\pm}$ in terms of the functions $\phi_{\pm}$ are obtained directly from corollary 7.19. ∎

Example 9.3. Consider the prototypical case discussed in the introduction. For this case the functions $\phi_{\pm}$ are given by the following formulas, valid for any $\zeta \in H$:

$$\left.\begin{aligned} \phi_{\pm}(\zeta) \equiv \pm(2\rho)^{-1}, \qquad \rho \equiv (b^2 - a\overline{a})^{1/2}, \\ b \equiv g(\zeta, \overline{\zeta}), \qquad a \equiv g(\zeta,\zeta), \qquad \overline{a} \equiv g(\overline{\zeta},\overline{\zeta}). \end{aligned}\right\} \tag{9.21}$$

Here $g(\cdot,\cdot)$ is a complex symmetric bilinear form on the complex vector space $V_{\mathbb{C}}$, which is real and definite when restricted to the real space V.

Note that, via the Cauchy–Schwarz inequality, we have $\rho \geqslant 0$ and $\rho = 0$ if and only if $\zeta \wedge \overline{\zeta} = 0$, so we have $\rho > 0$ on the space H and the functions $f_{\pm}$ are globally defined and smooth on the space H, as required. Also the functions $f_{\pm}$ have homogeneity degree $(-1,-1)$ in the variables $(\zeta, \overline{\zeta})$.

Next we have the following derivatives:

$$\pm 2\rho^3 \partial_{\zeta}\phi_{\pm} = \overline{a} g(\zeta,\cdot) - b g(\overline{\zeta},\cdot); \tag{9.22}$$

$$\begin{aligned} \pm 2\rho^5 \partial_{\overline{\zeta}} \otimes \partial_{\zeta}\phi_{\pm} &= 3[a g(\overline{\zeta},\cdot) - b g(\zeta,\cdot)] \otimes [\overline{a} g(\zeta,\cdot) - b g(\overline{\zeta},\cdot)] \\ &\quad + \rho^2[2g(\overline{\zeta},\cdot)\otimes g(\zeta,\cdot) - g(\zeta,\cdot)\otimes g(\overline{\zeta},\cdot) - b g(\cdot,\cdot)]; \end{aligned} \tag{9.23}$$

$$\begin{aligned} \pm 2\rho^5 \partial_{\overline{\zeta}} \wedge \partial_{\zeta}\phi_{\pm} &= 3[a g(\overline{\zeta},\cdot) - b g(\zeta,\cdot)] \wedge [\overline{a} g(\zeta,\cdot) - b g(\overline{\zeta},\cdot)] + 3\rho^2 g(\overline{\zeta},\cdot)\wedge g(\zeta,\cdot) \\ &= 3(a\overline{a} - b2 + \rho^2) g(\overline{\zeta},\cdot)\wedge g(\zeta,\cdot) = 0. \end{aligned} \tag{9.24}$$

So the functions $\phi_{\pm}$ obey the required field equations.

Next equation (9.16) gives the following differential equations, determining the quantities $\chi_{\pm}$:

$$\pm 2\rho^3 \partial_{\overline{\zeta}}\chi_{\pm}(\zeta) = \overline{a} g(\zeta,\cdot) - b g(\overline{\zeta},\cdot). \tag{9.25}$$

Solving equation (9.25) gives the following expression for the functions $\chi_{\pm}$:

$$\pm 2\chi_{\pm} = -\overline{a}\rho^{-1}(b+\rho)^{-1}. \tag{9.26}$$

Note that $b > 0$ if $\zeta \neq 0$, so that the functions $\chi_{\pm}$ are well defined if $\zeta \wedge \overline{\zeta} \neq 0$, so the functions $\chi_{\pm}$ are smooth and globally defined on the space H.

One computes the quantities $\pm\partial_{\overline{\zeta}}\chi_{\pm}$ as follows:

$$\begin{aligned} \pm 2(b+\rho)^2\rho^3 \partial_{\overline{\zeta}}\chi_{\pm} &= \tfrac{1}{2}\overline{a}(b+\rho)\partial_{\overline{\zeta}}\rho^2 - \rho^2(b+\rho)\partial_{\overline{\zeta}}\overline{a} + \overline{a}(\rho^2\partial_{\overline{\zeta}} b + \tfrac{1}{2}\rho\partial_{\overline{\zeta}}\rho^2) \\ &= -\overline{a}(b+2\rho)[a g(\overline{\zeta},\cdot) - b g(\zeta,\cdot)] - 2\rho^2(b+\rho) g(\overline{\zeta},\cdot) + \overline{a}\rho^2 g(\zeta,\cdot) \\ &= \overline{a} g(\zeta,\cdot)(b^2 + \rho^2 + 2b\rho) - b g(\overline{\zeta},\cdot)[\overline{a}a + 2\rho^2 + 2b\rho] \\ &= [\overline{a} g(\zeta,\cdot) - b g(\overline{\zeta},\cdot)](b+\rho)^2. \end{aligned} \tag{9.27}$$

So equation (9.25) is satisfied, as required.

Next by equation (9.18), we have the following expressions for the functions $g_\pm$, valid for any $\zeta \in H$:

$$g_\pm(\zeta) = \zeta\chi_\pm(\zeta) + \overline{\zeta}\phi_\pm(\zeta) = \pm(2\rho)^{-1}(\overline{\zeta} - \zeta\overline{a}(b+\rho)^{-1}). \tag{9.28}$$

Put $\zeta = v + \mathrm{i}tw$, where v and w are real and linearly independent and t is real and non-zero. Also put $p \equiv g(v,v)$, $q \equiv g(v,w)$, $r \equiv g(w,w)$ and $s \equiv \sqrt{(pr-q^2)}$. Then $s > 0$, by the Cauchy–Schwarz inequality.

We have $a = p + 2\mathrm{i}tq - t^2r$, $b = p + t^2r$ and

$$\rho^2 = (p+t^2r)^2 - (p+2\mathrm{i}tq - t^2r)(p - 2\mathrm{i}tq - t^2r) = 4t^2s^2.$$

So we find that $\rho = 2|t|s$. Put $\epsilon(t) \equiv t/|t|$, for any real non-zero t.

Inserting these relations into equation (9.28), we get

$$\begin{aligned} \pm 2g_\pm(v+\mathrm{i}tw) &= |t|^{-1}[2s(p+2|t|s+t^2r)]^{-1}[v(b+\rho-\overline{a}) - \mathrm{i}tw(b+\rho+\overline{a})] \\ &= [s(p+2|t|s+t^2r)]^{-1} \\ &\quad \times [v(s+\mathrm{i}q\epsilon(t)+|t|r) - \mathrm{i}w(p\epsilon(t)+t(s-\mathrm{i}\epsilon(t)q)]. \end{aligned} \tag{9.29}$$

We take the real part of equation (9.29) and then take the limit as $t \to 0$. Using equation (9.6), this gives the following relation, valid for any $v \in V$:

$$vf(v) = vp^{-1} = vg(v,v)^{-1}. \tag{9.30}$$

So we have $f(v) = g(v,v)^{-1}$, for any $v \in V$ and we find that the function f coincides with the function f_g, of the introduction, as expected.

Note that by equation (9.8), we have the relation,

$$\varPhi(f_g)[m(\zeta^+ \wedge \zeta^-)] = \rho^{-1}(\mathrm{i}\zeta \wedge \overline{\zeta}). \tag{9.31}$$

Putting $\zeta = v + \mathrm{i}w$, with $v \in V$ and $w \in V$, this gives the formula,

$$\varPhi(f_g)(m(v\wedge w)) = 2\rho^{-1}(v \wedge w). \tag{9.32}$$

Also we have the relation,

$$\begin{aligned} \rho^2 = b^2 - a\overline{a} &= [g(v,v)+g(w,w)]^2 - [g(v,v)-g(w,w)]^2 - 4g(v,w)^2 \\ &= 4[g(v,v)g(w,w) - g(v,w)^2]. \end{aligned} \tag{9.33}$$

So we get the following formulas, valid for all linearly independent vectors v and w in the vector space V:

$$\left.\begin{aligned} \varPhi(f_g)(m(v\wedge w)) &= v\wedge w\phi(f_g)(v,w), \\ \phi(f_g)(v,w) &\equiv [g(v,v)g(w,w) - g(v,w)^2]^{-1/2}. \end{aligned}\right\} \tag{9.34}$$

These formulas are in exact agreement with the formulas of the introduction.

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